Solution Set 13

Problem 1: Optional stopping theorem

Let $(S_n,\, n \in \mathbb{N})$ be the simple symmetric random walk, $(\mathcal{F}_n,\, n \in \mathbb{N})$ be its natural filtration and

$$T = \inf\{n \geq 1 \,:\, S_n \geq a \quad \text{or} \quad S_n \leq -b\},$$

where a, b are positive integers.

a) Show that T is a stopping time with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

b) Use the optional stopping theorem to compute $\mathbb{P}(\{S_T = a\})$.

Let now $(M_n,\, n \in \mathbb{N})$ be defined as $M_n = S_n^2 - n$, for all $n \in \mathbb{N}$.

c) Show that the process $(M_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

d) Apply the optional stopping theorem to compute $\mathbb{E}(T)$.

Remark: Even though T is an unbounded stopping time, the optional stopping theorem applies both in parts b) and d). Notice that the theorem would *not* apply if one would consider the stopping time: $T' = \inf\{n \geq 1 \,:\, S_n \geq a\}$.

Solution a) For every $n \geq 0$, it holds that

$$\{T = n\} = \left(\cap_{j=0}^{n-1} \{-b < S_j < a\}\right) \bigcap \left(\{S_n \leq -b\} \cup \{S_n \geq a\}\right)$$

and as all these events belong to $\mathcal{F}_n$, so does $\{T = n\}$.

b) As a and b are positive integers, S_T can only take values in $\{-b, a\}$, so we obtain by the optional stopping theorem:

$$0 = \mathbb{E}(S_0) = \mathbb{E}(S_T) = a\,\mathbb{P}(S_T = a) - b\,\mathbb{P}(S_T = -b) = a\,\mathbb{P}(S_T = a) - b\,(1 - \mathbb{P}(\{S_T = a\}))$$

which implies that $\mathbb{P}(\{S_T = a\}) = \frac{b}{a+b}$ [quite a "logical" result, if you think about it].

c) For $n \geq 0$, let us compute

$$\begin{aligned} \mathbb{E}(M_{n+1}|\mathcal{F}_n) &= \mathbb{E}(S_{n+1}^2 - (n+1)|\mathcal{F}_n) = \mathbb{E}(S_n^2 + 2S_n X_{n+1} + X_{n+1}^2|\mathcal{F}_n) - (n+1) \\ &= S_n^2 + 2S_n\,\mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) - (n+1) = S_n^2 + 0 + 1 - (n+1) = S_n^2 - n = M_n \end{aligned}$$

which proves the claim.

d) Applying again the optional stopping theorem, we obtain

$$0 = \mathbb{E}(M_0) = \mathbb{E}(M_T) = \mathbb{E}(S_T^2 - T) = \mathbb{E}(S_T^2) - \mathbb{E}(T)$$

So by part b), we obtain

$$\mathbb{E}(T) = \mathbb{E}(S_T^2) = a^2\,\mathbb{P}(\{S_T = a\}) + b^2\,\mathbb{P}(\{S_T = -b\}) = a^2\,\frac{b}{a+b} + b^2\,\frac{a}{a+b} = ab$$

Problem 2: Martingale convergence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Let $U \sim \mathcal{U}([-1,+1])$ be a random variable independent of $\mathcal{G}$ and M be a positive, integrable and $\mathcal{G}$-measurable random variable.

a) Compute the function $\psi : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\psi(M) = \mathbb{E}(|M+U| \mid \mathcal{G})$$

Let now $(U_n,\, n \geq 1)$ be a sequence of i.i.d. $\sim \mathcal{U}([-1,+1])$ random variables, all defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(U_1, \ldots, U_n)$, $n \geq 1$. Let finally $(M_n,\, n \geq 1)$ be the process defined recursively as

$$M_0 = 0, \quad M_{n+1} = |M_n + U_{n+1}|, \quad n \in \mathbb{N}$$

b) Show that the process $(M_n,\, n \in \mathbb{N})$ is a submartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

c) Compute the unique predictable and increasing process $(A_n,\, n \in \mathbb{N})$ such that the process $(M_n - A_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n, ,\, n \in \mathbb{N})$.

d) Is it true that the process $(M_n^2,\, n \in \mathbb{N})$ is also a submartingale with respect to $(\mathcal{F}_n, ,\, n \in \mathbb{N})$? Justify your answer.

e) Determine the value of $c > 0$ such that the process $(N_n = M_n^2 - cn, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

f) Does there exist a random variable M_∞ such that $M_n \underset{n\to\infty}{\to} M_\infty$ almost surely? (Again, no formal justification required here; an intuitive argument will do.)

Solution a) Let us compute the function $\psi(m) = \mathbb{E}(|m+U|) = \frac{1}{2}\int_{-1}^{1} |m+u|\, du$. If $m \geq 1$, then

$$\psi(m) = \mathbb{E}(m+U) = m$$

while if $0 \leq m < 1$, then

$$\psi(m) = \frac{1}{2}\left(\int_{-1}^{-m} (-m-u)\, du + \int_{-m}^{+1} (m+u)\, du\right) = \frac{1+m^2}{2}$$

b) By what was computed in part a), we see that for $m \geq 0$, $\psi(m) \geq m$, so

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = \mathbb{E}(|M_n + U_{n+1}| \,|\, \mathcal{F}_n) = \psi(M_n) \geq M_n$$

c)

$$A_{n+1} - A_n = \mathbb{E}(M_{n+1} - M_n | \mathcal{F}_n) = \psi(M_n) - M_n = \begin{cases} 0 & \text{if } M_n \geq 1 \\ \frac{1+M_n^2}{2} - M_n = \frac{(1-M_n)^2}{2} & \text{if } M_n < 1 \end{cases}$$

so $A_n = \frac{1}{2}\sum_{j=1}^{n-1} (1-M_j)^2\, 1_{\{M_j < 1\}}$.

d) Yes:

$$\mathbb{E}(M_{n+1}^2 \,|\, \mathcal{F}_n) = \mathbb{E}((M_n + U_{n+1})^2 | \mathcal{F}_n) = M_n^2 + 2M_n \mathbb{E}(U_{n+1}) + \mathbb{E}(U_{n+1}^2) = M_n^2 + \frac{1}{3} \geq M_n^2$$

e) By the previous computation, we deduce that $c = \frac{1}{3}$.

f) No. M is a positive submartingale behaving like a random walk with uniform increments on the positive axis, and being reflected when it gets close to 0. It will not converge anywhere.

Problem 3: Recursive martingale convergence

Let $0 < p < 1$ and $M = (M_n,\, n \in \mathbb{N})$ be the process defined recursively as

$$M_0 = x \in]0,1[, \quad M_{n+1} = \begin{cases} p\, M_n, & \text{with probability } 1 - M_n \\ \\ (1-p) + pM_n, & \text{with probability } M_n \end{cases}$$

and $(\mathcal{F}_n,\, n \in \mathbb{N})$ be the filtration defined as $\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$, $n \in \mathbb{N}$.

a) For what value(s) of $0 < p < 1$ is the process M is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$? Justify your answer.

b) In the case(s) M is a martingale, compute $\mathbb{E}(M_{n+1}\,(1 - M_{n+1}) \,|\, \mathcal{F}_n)$ for $n \in \mathbb{N}$.

c) Deduce the value of $\mathbb{E}(M_n\,(1 - M_n))$ for $n \in \mathbb{N}$.

d) Does there exist a random variable M_∞ such that

(i) $M_n \underset{n\to\infty}{\to} M_\infty$ a.s. ? (ii) $M_n \underset{n\to\infty}{\overset{L^2}{\to}} M_\infty$? (iii) $\mathbb{E}(M_\infty|\mathcal{F}_n) = M_n,\ \forall n \in \mathbb{N}$?

e) What can you say more about M_∞? (No formal justification required here; an intuitive argument will do.)

Solution a) Observe first that for every value of $0 < p < 1$, we have $0 < M_{n+1} < 1$ whenever $0 < M_n < 1$ (and $M_0 = x \in]0,1[$ by assumption), so that it makes sense to consider both M_n and $1 - M_n$ as probabilities at every step. Let us then compute:

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = pM_n\,(1 - M_n) + ((1-p) + pM_n)\,M_n = M_n$$

for every $0 < p < 1$!

b) Again, let us compute:

$$\begin{aligned}\mathbb{E}(M_{n+1}\,(1 - M_{n+1}) \,|\, \mathcal{F}_n) &= pM_n\,(1 - pM_n)\,(1 - M_n) + ((1-p) + pM_n)\,(1 - (1-p) - pM_n)\,M_n \\ &= pM_n\,(1 - pM_n)\,(1 - M_n) + ((1-p) + pM_n)\,p(1 - M_n)\,M_n = p(2-p)\,M_n\,(1 - M_n)\end{aligned}$$

c) Therefore, we obtain by induction:

$$\mathbb{E}(M_n\,(1 - M_n)) = (p(2-p))^n\, x(1-x)$$

which converges to 0 as n gets large (as $p(2-p) < 1$ for every $0 < p < 1$).

d) Because the martingale M is bounded, the answer is yes to all three questions.

e) The answer obtained in c) suggests that M_n converges either to 0 or 1 as $n \to \infty$, which turns out to be the case. Moreover, as seen in class:

$$\mathbb{P}(\{M_\infty = 1\}) = \mathbb{E}(M_\infty) = \mathbb{E}(M_0) = x, \quad \text{so} \quad \mathbb{P}(\{M_\infty = 0\}) = 1 - x$$