Problem Set 13

Problem 1: Optional stopping theorem

Let $(S_n,\, n \in \mathbb{N})$ be the simple symmetric random walk, $(\mathcal{F}_n,\, n \in \mathbb{N})$ be its natural filtration and

$$T = \inf\{n \geq 1 \,:\, S_n \geq a \quad \text{or} \quad S_n \leq -b\},$$

where a, b are positive integers.

a) Show that T is a stopping time with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

b) Use the optional stopping theorem to compute $\mathbb{P}(\{S_T = a\})$.

Let now $(M_n,\, n \in \mathbb{N})$ be defined as $M_n = S_n^2 - n$, for all $n \in \mathbb{N}$.

c) Show that the process $(M_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

d) Apply the optional stopping theorem to compute $\mathbb{E}(T)$.

Remark: Even though T is an unbounded stopping time, the optional stopping theorem applies both in parts b) and d). Notice that the theorem would *not* apply if one would consider the stopping time: $T' = \inf\{n \geq 1 \,:\, S_n \geq a\}$.

Problem 2: Martingale convergence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Let $U \sim \mathcal{U}([-1,+1])$ be a random variable independent of $\mathcal{G}$ and M be a positive, integrable and $\mathcal{G}$-measurable random variable.

a) Compute the function $\psi : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\psi(M) = \mathbb{E}(|M+U| \mid \mathcal{G})$$

Let now $(U_n,\, n \geq 1)$ be a sequence of i.i.d. $\sim \mathcal{U}([-1,+1])$ random variables, all defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(U_1, \ldots, U_n)$, $n \geq 1$. Let finally $(M_n,\, n \geq 1)$ be the process defined recursively as

$$M_0 = 0, \quad M_{n+1} = |M_n + U_{n+1}|, \quad n \in \mathbb{N}$$

b) Show that the process $(M_n,\, n \in \mathbb{N})$ is a submartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

c) Compute the unique predictable and increasing process $(A_n,\, n \in \mathbb{N})$ such that the process $(M_n - A_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,,\, n \in \mathbb{N})$.

d) Is it true that the process $(M_n^2,\, n \in \mathbb{N})$ is also a submartingale with respect to $(\mathcal{F}_n,,\, n \in \mathbb{N})$? Justify your answer.

e) Determine the value of $c > 0$ such that the process $(N_n = M_n^2 - cn, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

f) Does there exist a random variable M_∞ such that $M_n \underset{n\to\infty}{\to} M_\infty$ almost surely? (Again, no formal justification required here; an intuitive argument will do.)

Problem 3: Recursive martingale convergence

Let $0 < p < 1$ and $M = (M_n,\, n \in \mathbb{N})$ be the process defined recursively as

$$M_0 = x \in \,]0,1[, \quad M_{n+1} = \begin{cases} p\, M_n, & \text{with probability } 1 - M_n \\ \\ (1-p) + pM_n, & \text{with probability } M_n \end{cases}$$

and $(\mathcal{F}_n,\, n \in \mathbb{N})$ be the filtration defined as $\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$, $n \in \mathbb{N}$.

a) For what value(s) of $0 < p < 1$ is the process M is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$? Justify your answer.

b) In the case(s) M is a martingale, compute $\mathbb{E}(M_{n+1}\,(1 - M_{n+1}) \,|\, \mathcal{F}_n)$ for $n \in \mathbb{N}$.

c) Deduce the value of $\mathbb{E}(M_n\,(1 - M_n))$ for $n \in \mathbb{N}$.

d) Does there exist a random variable M_∞ such that

(i) $M_n \underset{n\to\infty}{\to} M_\infty$ a.s. ? (ii) $M_n \overset{L^2}{\underset{n\to\infty}{\to}} M_\infty$? (iii) $\mathbb{E}(M_\infty|\mathcal{F}_n) = M_n,\ \forall n \in \mathbb{N}$?

e) What can you say more about M_∞? (No formal justification required here; an intuitive argument will do.)