Solution Set 12

Problem 1: Increasing martingale

a) Let $(M_n,\, n \in \mathbb{N})$ be an *increasing* martingale, that is, $M_{n+1} \geq M_n$ a.s. for all $n \in \mathbb{N}$. Show that $M_n = M_0$ a.s., for all $n \in \mathbb{N}$.

b) Let $(M_n,\, n \in \mathbb{N})$ be a square-integrable martingale such that $(M_n^2,\, n \in \mathbb{N})$ is also a martingale. Show that $M_n = M_0$ a.s., for all $n \in \mathbb{N}$.

Solution a) We know that $M_{n+1} - M_n \geq 0$ a.s., for all $n \geq 0$, and since M is a martingale, we also know that $\mathbb{E}(M_{n+1} - M_n) = 0$ for all $n \geq 0$, so $M_{n+1} = M_n$ a.s. for all $n \geq 0$, i.e. $M_n = M_0$ a.s. for all $n \geq 0$.

b) Let us compute, for $n \geq 0$,

$$\begin{aligned}
\mathbb{E}((M_{n+1} - M_n)^2) &= \mathbb{E}(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2) = \mathbb{E}(\mathbb{E}(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2|\mathcal{F}_n)) \\
&= \mathbb{E}(\mathbb{E}(M_{n+1}^2|\mathcal{F}_n) - 2\mathbb{E}(M_{n+1}|\mathcal{F}_n)M_n + M_n^2) = \mathbb{E}(M_n^2 - 2M_n^2 + M_n^2) = 0
\end{aligned}$$

where we have used the assumption that $\mathbb{E}(M_{n+1}^2|\mathcal{F}_n) = M_n^2$. Therefore, $M_n = M_0$ a.s. for all $n \geq 0$.

Problem 2: Recursive martingale

Let $0 < p < 1$ and $x > 0$ be fixed real numbers and $(X_n,\, n \in \mathbb{N})$ be the process defined recursively as

$$X_0 = x, \quad X_{n+1} = \begin{cases} X_n^2 + 1 & \text{with probability } p \\ X_n/2 & \text{with probability } 1-p \end{cases} \quad \text{for } n \in \mathbb{N}$$

a) What *minimal* condition on $0 < p < 1$ guarantees that the process X is a submartingale (with respect to its natural filtration)? Justify your answer.

Hint: The inequality $a^2 + b^2 \geq 2ab$ may be useful here.

b) For the values of p respecting the condition found in part a), derive a lower bound on $\mathbb{E}(X_n)$.

Hint: Proceed recursively.

c) Does there exist a value of $0 < p < 1$ such that the process X is a martingale? a supermartingale? Again, justify your answer.

Solution a) Let $(\mathcal{F}_n,\, n \geq 1)$ denote natural filtration of $(X_n, n \in \mathbb{N})$. Using the hint, we find

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2 + 1) + (1-p)\,\frac{X_n}{2} \geq 2p\,X_n + (1-p)\,\frac{X_n}{2} = \frac{3p+1}{2}\,X_n \geq X_n$$

if $p \geq 1/3$. This condition turns out to be the minimal one. Indeed, it can always happen that X_n gets arbitrarily close to the value 1. In this case,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2+1) + (1-p)\,\frac{X_n}{2} \sim 2p + \frac{1-p}{2} = \frac{3p+1}{2}$$

which is strictly less than 1 if $p < \frac{1}{3}$.

b) When $p \geq \frac{1}{3}$, we have

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(\mathbb{E}(X_{n+1}|\mathcal{F}_n)) \geq \mathbb{E}\left(\frac{3p+1}{2}\,X_n\right) = \frac{3p+1}{2}\,\mathbb{E}(X_n)$$

so

$$\mathbb{E}(X_n) \geq \left(\frac{3p+1}{2}\right)^n x$$

c) No. The justification for this is the following: it can always happen that X_n follows the "down" path so as to get arbitrarily close to the value zero. In this case,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2+1) + (1-p)\,\frac{X_n}{2} \sim p > X_n$$

so the supermartingale property (and therefore also the martingale property) fails to hold, for any fixed value of $p > 0$.

Problem 3: Martingale transform

Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$. Let also $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \geq 1$ and let $(H_n,\, n \in \mathbb{N})$ be a predictable process with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\exists K_n > 0$ with $|H_n(\omega)| \leq K_n$ for all $\omega \in \Omega$. Let finally

$$G_0 = 0 \quad \text{and} \quad G_n = \sum_{j=1}^n H_j X_j, \quad n \geq 1.$$

From the course, we know that the process G is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

a) Under the assumptions made, is it possible that $\mathbb{E}(H_j X_j) > 0$ for some j? Explain!

b) Find the unique predictable and increasing process $(A_n,\, n \in \mathbb{N})$ such that the process $(G_n^2 - A_n,\, n \in \mathbb{N})$ is also a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

From now on, consider the particular case where $H_n(\omega) \in \{-1, +1\}$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$.

c) Compute the process A in this particular case.

d) Let $a \geq 1$ be an integer and let $T = \inf\{n \geq 1 \,:\, |G_n| \geq a\}$. Compute $\mathbb{E}(T)$ [no full justification required here].

Solution a) No. H_j is $\mathcal{F}_{j-1}$-measurable, while X_j is independent of $\mathcal{F}_{j-1}$, so

$$\mathbb{E}(H_j X_j) = \mathbb{E}(H_j)\,\mathbb{E}(X_j) = \mathbb{E}(H_j)\,0 = 0$$

b) We have

$$\begin{aligned} A_{n+1} - A_n &= \mathbb{E}(G_{n+1}^2|\mathcal{F}_n) - G_n^2 = \mathbb{E}((G_n + H_{n+1}X_{n+1})^2|\mathcal{F}_n) - G_n^2 \\ &= G_n^2 + 2G_n H_{n+1}\mathbb{E}(X_{n+1}) + H_{n+1}^2\mathbb{E}(X_{n+1}^2) - G_n^2 = H_{n+1}^2 \end{aligned}$$

so $A_0 = 0$ and $A_n = \sum_{j=1}^n H_j^2$.

c) $H_j^2 = 1$ for all j, so $A_n = n$.

d) Here, the idea is to use the optional stopping theorem with the martingale $(G_n^2 - n,\, n \in \mathbb{N})$, which gives

$$\mathbb{E}(G_T^2 - T) = \mathbb{E}(G_0^2 - 0) = 0, \quad \text{so} \quad \mathbb{E}(T) = \mathbb{E}(G_T^2) = a^2$$

Unfortunately, a full justification of the use of the theorem is impossible here using only the tools that you have learned in class, because the martingale $(G_n^2 - n,\, n \in \mathbb{N})$ is not bounded from below until the (unbounded) time T.