Problem Set 12

Problem 1: Increasing martingale

a) Let $(M_n,\, n \in \mathbb{N})$ be an *increasing* martingale, that is, $M_{n+1} \geq M_n$ a.s. for all $n \in \mathbb{N}$. Show that $M_n = M_0$ a.s., for all $n \in \mathbb{N}$.

b) Let $(M_n,\, n \in \mathbb{N})$ be a square-integrable martingale such that $(M_n^2,\, n \in \mathbb{N})$ is also a martingale. Show that $M_n = M_0$ a.s., for all $n \in \mathbb{N}$.

Problem 2: Recursive martingale

Let $0 < p < 1$ and $x > 0$ be fixed real numbers and $(X_n,\, n \in \mathbb{N})$ be the process defined recursively as

$$X_0 = x, \quad X_{n+1} = \begin{cases} X_n^2 + 1 & \text{with probability } p \\ X_n/2 & \text{with probability } 1-p \end{cases} \quad \text{for } n \in \mathbb{N}$$

a) What *minimal* condition on $0 < p < 1$ guarantees that the process X is a submartingale (with respect to its natural filtration)? Justify your answer.

Hint: The inequality $a^2 + b^2 \geq 2ab$ may be useful here.

b) For the values of p respecting the condition found in part a), derive a lower bound on $\mathbb{E}(X_n)$.

Hint: Proceed recursively.

c) Does there exist a value of $0 < p < 1$ such that the process X is a martingale? a supermartingale? Again, justify your answer.

Problem 3: Martingale transform

Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$. Let also $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \geq 1$ and let $(H_n,\, n \in \mathbb{N})$ be a predictable process with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\exists K_n > 0$ with $|H_n(\omega)| \leq K_n$ for all $\omega \in \Omega$. Let finally

$$G_0 = 0 \quad \text{and} \quad G_n = \sum_{j=1}^{n} H_j X_j, \quad n \geq 1.$$

From the course, we know that the process G is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

a) Under the assumptions made, is it possible that $\mathbb{E}(H_j X_j) > 0$ for some j? Explain!

b) Find the unique predictable and increasing process $(A_n,\, n \in \mathbb{N})$ such that the process $(G_n^2 - A_n,\, n \in \mathbb{N})$ is also a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

From now on, consider the particular case where $H_n(\omega) \in \{-1, +1\}$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$.

c) Compute the process A in this particular case.

d) Let $a \geq 1$ be an integer and let $T = \inf\{n \geq 1 : |G_n| \geq a\}$. Compute $\mathbb{E}(T)$ [no full justification required here].