Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 14

No submission!

Definition. Let M be a smooth manifold with or without boundary.

(a) A *curve segment in* M is defined to be a continuous curve $\gamma\colon [a,b] \to M$ whose domain is a compact interval. It is a *smooth curve segment in* M if it is smooth when $[a,b]$ is considered as a manifold with boundary (or, equivalently, if γ has an extension to a smooth curve defined in a neighborhood of each endpoint). It is a *piecewise smooth curve segment in* M if there exists a finite partition $a_0 = a < a_1 < \cdots < a_{k-1} < a_k = b$ of $[a,b]$ such that $\gamma|_{[a_{i-1},a_i]}$ is smooth[1] for every $1 \leq i \leq k$.

(b) Let ω be a smooth covector field on M. If $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment, then *the line integral of* ω *over* γ is defined to be the real number

$$\int_\gamma \omega := \sum_{i=1}^k \int_{[a_{i-1},a_i]} \gamma^*\omega,$$

where $[a_{i-1},a_i]$, $1 \leq i \leq k$, are subintervals of $[a,b]$ on which γ is smooth. If t denotes the standard coordinate on $\mathbb{R}$, then the smooth covector field $\omega_i := \gamma^*\omega = \left(\gamma|_{[a_{i-1},a_i]}\right)^*\omega$ on $[a_{i-1},a_i]$ can be written as $\omega_i = f_i(t)\,dt$ for some smooth function $f_i\colon [a_{i-1},a_i] \to \mathbb{R}$, so the integral of ω_i over $[a_{i-1},a_i]$ is given by

$$\int_{[a_{i-1},a_i]} \omega_i = \int_{a_{i-1}}^{a_i} f_i(t)\,dt.$$

Therefore,

$$\int_\gamma \omega = \sum_{i=1}^k \int_{a_{i-1}}^{a_i} f_i(t)\,dt.$$

[1] Continuity of γ means that $\gamma(t)$ approaches the same value as t approaches any of the points a_i (other than a_0 or a_k) from the left or the right. Smoothness of γ in each subinterval means that γ has one-sided velocity vectors at each such a_i when approaching from the left or the right, but these one-sided velocities need not be equal.

Exercise 1 (*Properties of line integrals*):
Let M be a smooth manifold with or without boundary. Let $\gamma\colon [a,b] \to M$ be a piecewise smooth curve segment in M, and let ω, ω_1, $\omega_2 \in \mathfrak{X}^*(M)$. Prove the following assertions:

(a) For any $c_1, c_2 \in \mathbb{R}$ we have

$$\int_\gamma \big(c_1\omega_1 + c_2\omega_2\big) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

(b) If γ is a constant map, then

$$\int_\gamma \omega = 0.$$

(c) If $\gamma_1 := \gamma|_{[a,c]}$ and $\gamma_2 := \gamma|_{[c,b]}$, where $a, b, c \in \mathbb{R}$ with $a < c < b$, then

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

(d) If $F\colon M \to N$ is any smooth map and if $\eta \in \mathfrak{X}^*(N)$, then

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Definition. Let M be a smooth manifold with or without boundary. If $\gamma\colon [a,b] \to M$ and $\widetilde{\gamma}\colon [c,d] \to M$ are piecewise smooth curve segments in M, then we say that $\widetilde{\gamma}$ is a *reparametrization* of γ if $\widetilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi\colon [c,d] \to [a,b]$. If φ is an increasing function, then we say that $\widetilde{\gamma}$ is a *forward reparametrization*, while if φ is a decreasing function, then we say that $\widetilde{\gamma}$ is a *backward reparametrization*. (More generally, with obvious modifications one can allow φ to be piecewise smooth.)

Exercise 2 (*Parameter independence of line integrals*):
Let M be a smooth manifold with or without boundary, $\omega \in \mathfrak{X}^*(M)$, and let γ be a piecewise smooth curve segment in M. Show that for any reparametrization $\widetilde{\gamma}$ of γ we have

$$\int_{\widetilde{\gamma}} \omega = \begin{cases} \displaystyle\int_\gamma \omega & \text{if } \widetilde{\gamma} \text{ is a forward reparametrization,} \\[2ex] \displaystyle -\int_\gamma \omega & \text{if } \widetilde{\gamma} \text{ is a backward reparametrization.} \end{cases}$$

Exercise 3:
Let M be a compact, connected, oriented, smooth n-manifold without boundary (i.e., $\partial M = \emptyset$), where $n \geq 1$, and let $\omega \in \Omega^{n-1}(M)$. Show that there exists a point $p \in M$ such that $(d\omega)_p = 0 \in \Lambda^n(T_p^*M)$.

Exercise 4:

(a) Let M be a smooth n-manifold (without boundary) and let $\omega \in \Omega^1(M)$.

(i) Let $\big(U, (x^i)\big)$ be a smooth coordinate chart for M, and write $\omega|_U = \sum_{i=1}^n \omega_i \, dx^i$ in this chart. Find an expression for the *exterior derivative* $d\omega \in \Omega^2(M)$ of ω in this chart (that is, an expression of $d\omega$ in terms of the natural basis induced in each fiber of $\Lambda^2(T^*M)$ by the given chart).

(ii) Deduce that ω is closed if and only if for every point $p \in M$ there exists a smooth coordinate chart $\big(U, (x^i)\big)$ such that $p \in U$ and

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j} \quad \text{for all } 1 \leq i, j \leq n,$$

where $\omega|_U = \sum_{i=1}^n \omega_i \, dx^i$ in this chart.

(b) Consider the smooth 1-forms

$$\omega = y\cos(xy)\, dx + x\cos(xy)\, dy \in \Omega^1(\mathbb{R}^2)$$

and

$$\eta = x\cos(xy)\, dx + y\cos(xy)\, dy \in \Omega^1(\mathbb{R}^2).$$

(i) Show that ω is closed and exact.

(ii) Show that η is neither closed nor exact.

(iii) Compute $\omega \wedge \eta$.

(c) Evaluate the line integral

$$\int_\gamma \omega,$$

where γ is the straight line segment from $(0, 0)$ to $(\sqrt{\pi}, \sqrt{\pi})$.

Exercise 5:
Consider the covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^3)$ given by

$$\omega = e^{y^2}\, dx + 2xye^{y^2}\, dy - 2z\, dz.$$

(a) Verify by direct computation that ω is closed.

(b) Using the fact that $\omega \in \mathfrak{X}^*(\mathbb{R}^3)$ is exact on the star-shaped set $\mathbb{R}^3$ (which follows from *Poincaré's lemma*), find a *potential* for ω, i.e., a function $f \in C^\infty(\mathbb{R}^3)$ such that $\omega = df$.

(c) Compute the line integral of ω along the smooth curve segment $\gamma\colon [0,1] \to \mathbb{R}^3,\ t \mapsto (t, t^2, t^3)$.