Problem Set 6
Signals

For the Exercise Session on Nov 26 — **Due: Tue, Dec 2, 10am, on Moodle**

1 Problem for Class

Problem 1: The Fourier matrix diagonalizes all circulant matrices.

The discrete Fourier transform (DFT) $\mathbf{X}$ of the vector $\mathbf{x}$ is given by

$$\mathbf{X} = W\mathbf{x} \quad \text{and} \quad \mathbf{x} = \frac{1}{N} W^H \mathbf{X}. \tag{1}$$

In this homework problem, you will prove that the Fourier matrix diagonalizes all *circulant* matrices.

(a) To cut the derivation into two simpler steps, we introduce an auxiliary matrix M, defined as

$$M = WA \quad = \quad W \underbrace{\begin{pmatrix} b_0 & b_{N-1} & b_{N-2} & b_{N-3} & \dots & b_1 \\ b_1 & b_0 & b_{N-1} & b_{N-2} & \dots & b_2 \\ b_2 & b_1 & b_0 & b_{N-1} & \dots & b_3 \\ b_3 & b_2 & b_1 & b_0 & \dots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N-1} & b_{N-2} & b_{N-3} & b_{N-4} & \dots & b_0 \end{pmatrix}}_{\text{This is a circulant matrix}}. \tag{2}$$

Let us denote the unitary DFT of the sequence $\{b_0, b_1, \dots, b_{N-1}\}$ by $\{B_0, B_1, \dots, B_{N-1}\}$. Write out the matrix M in terms of $\{B_0, B_1, \dots, B_{N-1}\}$. *Hint:* The first column of the matrix M is simply given by

$$W \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{N-1} \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{N-1} \end{pmatrix} \tag{3}$$

To find the second column, you will need to use some Fourier properties.

(b) Using the matrix M from above, compute the full matrix product

$$WAW^H \quad = \quad MW^H. \tag{4}$$

Hint: Handle every *row* of the matrix M separately. Define the vector $\mathbf{m}$ such that $\mathbf{m}^H$ is simply the first row of the matrix M. But the product $\mathbf{m}^H W^H$ is easily computed, recalling that $\mathbf{m}^H W^H = (W\mathbf{m})^H$.

2 The Homework

Problem 2:

(a) What is the parametric-form maximum entropy density $f(x)$ satisfying the two conditions

$$\mathbb{E}[X^8] = a \qquad \mathbb{E}[X^{16}] = b$$

(b) What is the maximum entropy density satisfying the condition

$$\mathbb{E}[X^8 + X^{16}] = a + b$$

(c) Which entropy is higher?

3 Additional Problems

Problem 3: Exponential Families and Maximum Entropy

Let $Y = X_1 + X_2$. Find the maximum entropy of Y under the constraint $\mathbb{E}[X_1^2] = P_1$, $\mathbb{E}[X_2^2] = P_2$:

(a) If X_1 and X_2 are independent.

(b) If X_1 and X_2 are allowed to be dependent.

Problem 4: Exponential Families and Maximum Entropy

For $t > 0$, consider a family of distributions supported on $[t, +\infty]$ such that $\mathbb{E}[\ln X] = \frac{1}{\alpha} + \ln t$, $\alpha > 0$.

1. What is the parametric form of a maximum entropy distribution satisfying the constraint on the support and the mean?
2. Find the exact form of the distribution.

Problem 5: Minimum-norm Solutions

In this problem, we consider an *underdetermined* system of linear equations, i.e., $A\mathbf{x} = \mathbf{b}$, where $A_{m\times n}$ is a "wide" matrix ($m < n$) and $\mathbf{b}$ is chosen such that a solution exists. As you know, in this case, there exist infinitely many solutions. Prove that the one solution $\mathbf{x}$ that has the minimum 2-norm can be expressed as

$$\mathbf{x}_{MN} \quad = \quad V\Sigma^{-1}U^H\mathbf{b}, \tag{5}$$

where, as usual, the SVD of $A = U\Sigma V^H$, and Σ^{-1} is the matrix Σ where all non-zero diagonal entries are inverted.

Hint: Clearly, A is not a full-rank matrix, and thus cannot be inverted. However, it might be possible to *construct* a matrix A' such that $A'\mathbf{x} = \mathbf{b}'$ has a solution, A is a submatrix of A' and $\mathbf{b}$ is a subvector of $\mathbf{b}'$. What will be the norm of $\mathbf{x}$ in such a case?

Problem 6: Exponential Families and Maximum Entropy 2

Find the maximum entropy density f, defined for $x \geq 0$, satisfying $\mathbb{E}[X] = \alpha_1$, $\mathbb{E}[\ln X] = \alpha_2$. That is, maximize $-\int f \ln f$ subject to $\int x f(x) dx = \alpha_1$, $\int (\ln x) f(x) dx = \alpha_2$, where the integral is over $0 \leq x < \infty$. What family of densities is this?