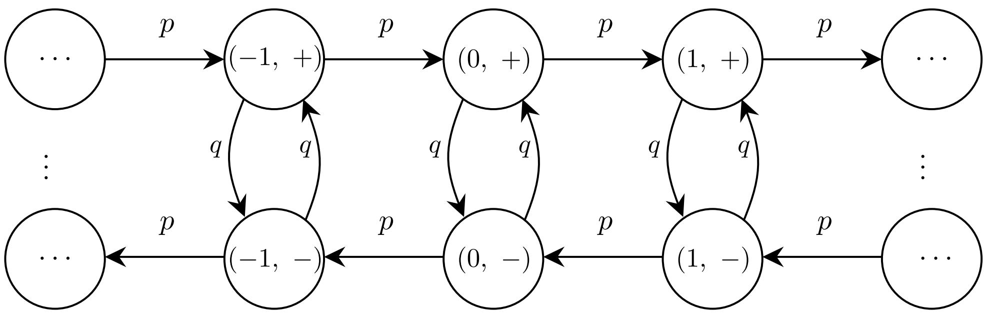
Midterm Exam Solution

Exercise 1. (24 points)

a)



b)

- $p = 0$: the equivalence classes are $\{(i, +), (i, -)\}$ for $i \in \mathbb{Z}$; that is, there are countably many equivalence classes.
- $p = 1$: each state $(i, u) \in S$ forms its own equivalence class $\{(i, u)\}$; again, the set of equivalence classes is countably infinite.
- $0 < p < 1$: all states S belong to a single equivalence class, i.e. the chain is irreducible.

c) The smallest n such that $p^{(n)}_{(0,+),(0,+)} > 0$ is $n = 2$: we use a path from $(0, +)$ to itself is going $(0, -)$ and back. Therefore, period cannot be greater than 2.

On the other hand, on the transition from (i, u), we either change the parity of i and keep the sign of the second element the same, or change the sign of u and keep the i and thus parity of the first element the same. Therefore, to get to the state with the same parity and with the same sign, we have to make even number of transitions of the first kind and even number of transitions of the second kind. Therefore, $p^{(n)}_{(0,+),(0,+)}$ can only be greater than 0 if n is even.

Thus, the periodicity of the state is 2.

Bonus (2 points). Consider each pair of states $(i, +)$ and $(i, -)$; if we treat them "together" as state i, the process resembles a symmetric random walk, which is recurrent, but with self-loops. In this structure, we would visit each state an infinite number of times. Going back to the original chain, as we visit each state an infinite number of times, visiting both $(i, +)$ and $(i, -)$ an infinite number of times seems more likely than just one of them.

d) For any $i \in \mathbb{Z}$ and any stationary distribution π, the following holds:

$$\pi_{i,-} = \pi_{i,+} \cdot q + \pi_{i+1,-} \cdot p$$

Thus, as $\pi_{j,-} = \pi_{j,+}$ for all $j \in \mathbb{Z}$, we can conclude that:

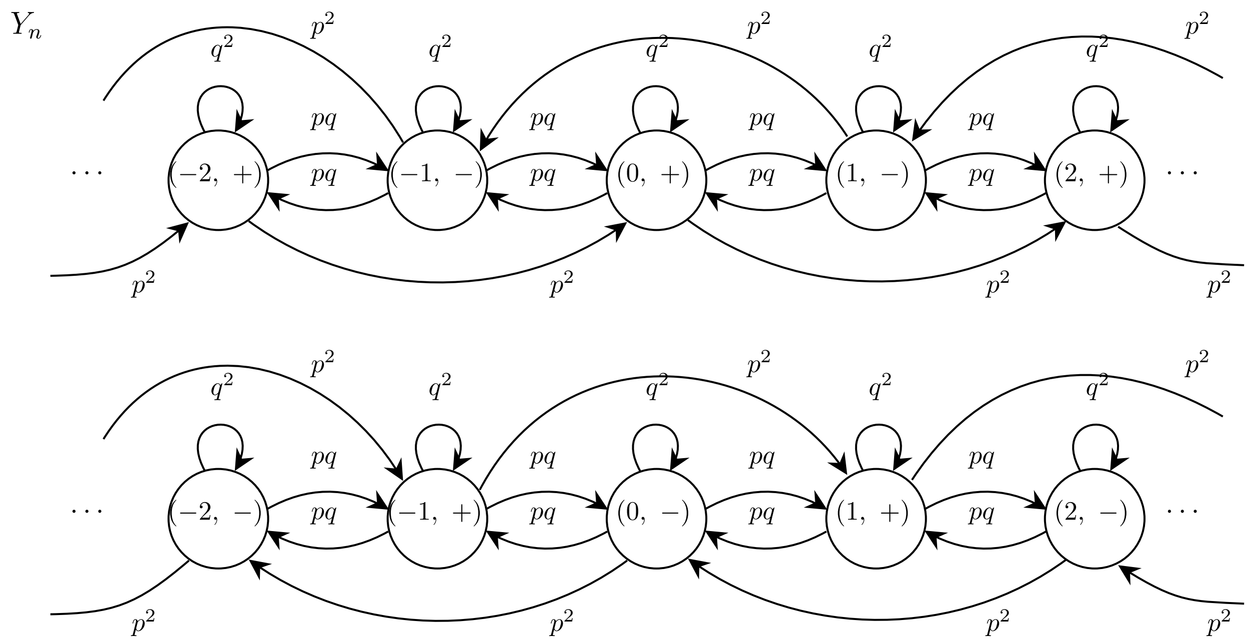
$$\pi_{i,+} = \pi_{i,+} \cdot q + \pi_{i+1,+} \cdot p \;\Rightarrow\; \pi_{i,+} p = \pi_{i+1,+} p \;\Rightarrow\; \pi_{i,+} = \pi_{i+1,+}.$$

Since this holds for all $i \in \mathbb{Z}$, we deduce that $\pi_{i,-} = \pi_{i,+} = \pi_{j,+} = \pi_{j,-}$ for any $i, j \in \mathbb{Z}$.

e) The chain is thus null-recurrent. Indeed, the chain is irreducible; if it were positive-recurrent, then by the theorem seen in class, it would admit a stationary distribution with $\pi_{i,u} > 0$ for all $(i, u) \in S$. Suppose $\pi_{0,+} = c > 0$. From d), the stationary distribution should be uniform, and thus the probabilities would sum up to:

$$\sum_{(i,u)\in S} \pi_{i,u} = \sum_{(i,u)\in S} c = +\infty$$

Hence, the chain does not admit a stationary distribution, and as we take the chain known to be recurrent, it must be null-recurrent.



f) Using the Chapman-Kolmogorov equations, we obtain:

$$\begin{aligned}
q_{(i,u),(j,v)} &= \sum_{(k,w)\in S} p_{(i,u),(k,w)} \cdot p_{(k,w),(j,v)} \\
&= p \cdot p_{(i+u,u),(j,v)} + q \cdot p_{(i,-u),(j,v)} \\
&= \begin{cases} p^2, & \text{if } j, v = i + 2u, u \\ pq, & \text{if } j, v = i \pm u, -u \\ q^2, & \text{if } j, v = i, u \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

g) There are two equivalence classes: (see the above graph)

- $C_0 := \{(2k, +), k \in \mathbb{Z}\} \cup \{(2k + 1, -), k \in \mathbb{Z}\} = \{(k, (-1)^k), k \in \mathbb{Z}\}$
- $C_1 := \{(2k, -), k \in \mathbb{Z}\} \cup \{(2k + 1, +), k \in \mathbb{Z}\} = \{(k, (-1)^{k+1}), k \in \mathbb{Z}\}$

Exercise 2 (24 points).

a) As $i - j$ takes values in $\{-4, -3, \dots, 3, 4\}$, its absolute value $|i - j|$ is in $\{0, \dots, 4\}$, and $d(i, j)$ can then only be in $S_D = \{0, 1, 2\}$.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

- $D_n = d(X_n, Y_n) = 0$ implies $X_n = Y_n$ and thus they evolve together and from there on, D_n stays 0.
- $D_n = d(X_n, Y_n) = 1$: with probability 1/2, X_n and Y_n evolve in the same direction, and the cyclic distance stays the same. When X_n and Y_n move toward each other – with probability 1/4 – the cyclic distance is again 1. When X_n and Y_n move in opposite directions – with probability 1/4 – the path between states becomes of length 3, while the alternative becomes of 2, hence the cyclic distance is 2.
- $D_n = d(X_n, Y_n) = 2$: like before, when X_n and Y_n evolve in the same direction – with probability 1/2 – the cyclic distance stays the same. Now, if X_n and Y_n move toward each other – with probability 1/4 – they will meet and distance becomes 0. Finally, if X_n and Y_n move in opposite ways – also with probability 1/4 – the alternative path ends up shorter, thus the cyclic distance becomes 1.

b) There are two equivalence classes: recurrent $\{0\}$ and transient $\{1, 2\}$.

c) If π is a stationary distribution, we have to assign the states from the transient class probability zero, i.e., $\pi_1 = \pi_2 = 0$, which leaves us with $\pi_0 = 1$; thus, it must be unique. Such a distribution is indeed a stationary one.

Now, as the 1 and 2 are transient states, we have $\sum_{n \geq 1} p_{11}^{(n)} < +\infty$ and $\sum_{n \geq 1} p_{22}^{(n)} < +\infty$, implying $\lim_{n \to \infty} p_{11}^{(n)} = \lim_{n \to \infty} p_{22}^{(n)} = 0$. Then,

$$\begin{aligned} p_{11}^{(n)} &= \frac{3}{4} p_{11}^{(n-1)} + \frac{1}{4} p_{21}^{(n-1)} \\ p_{22}^{(n)} &= \frac{1}{4} p_{12}^{(n-1)} + \frac{1}{2} p_{22}^{(n-1)} \end{aligned}$$

implies $\lim_{n \to \infty} p_{12}^{(n)} = \lim_{n \to \infty} p_{21}^{(n)} = 0$. As the last 2 columns of P^n converge to zeros, $\lim_{n \to \infty} P^n = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Thus, the stationary distribution $\pi = (1, 0, 0)$ is a limiting one.

Note: An alternative justification as for why is π also a limiting distribution comes from the ergodic theorem stated (even though it was only formally stated for an irreducible chain in the lectures).

d) To find the eigenvalues of P, we find the characteristic polynomial of P, so $\det(P - \lambda I)$. Using the cofactors formula, we obtain

$$\begin{aligned}
\det(P - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3/4-\lambda & 1/4 \\ 1/4 & 1/4 & 1/2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 3/4-\lambda & 1/4 \\ 1/4 & 1/2-\lambda \end{vmatrix} \\
&= (1-\lambda)\left(\left(\frac{3}{4}-\lambda\right)\left(\frac{1}{2}-\lambda\right) - \frac{1}{16}\right) = (1-\lambda)\left(\lambda^2 - \frac{5}{4}\lambda + \frac{5}{16}\right) \\
&= (1-\lambda)\left(\left(\lambda - \frac{5}{8}\right)^2 - \frac{5}{64}\right) = (1-\lambda)\left(\lambda - \frac{5-\sqrt{5}}{8}\right)\left(\lambda - \frac{5+\sqrt{5}}{8}\right)
\end{aligned}$$

Thus, $\lambda_0 = 1$, $\lambda_1 = \frac{5+\sqrt{5}}{8}$, and $\lambda_2 = \frac{5-\sqrt{5}}{8}$.

Note: Alternatively, we may use the fact that $\lambda_0 = 1$ (as P is a stochastic matrix), together with the fact that $\mathrm{Tr}(P) = 9/4$ and $\det(P) = 5/16$; this leads to a quadratic equation for λ_1, λ_2.

e) Let us compute

$$\begin{aligned}
\|\mu - \delta_0\|_{\mathrm{TV}} &= \frac{1}{2}\sum_{j \in S} |\mu_j - \delta_{0j}| = |\mu_0 - 1| + \sum_{j \in S : j \neq 0} \mu_j \\
&= \frac{1}{2}((1-\mu_0) + (1-\mu_0)) = 1 - \mu_0
\end{aligned}$$

f) From part e), we obtain

$$\mathbb{P}(X_n \neq Y_n) = \mathbb{P}(D_n \neq 0) = \left\|\pi^{(n)} - \pi\right\|_{\mathrm{TV}}$$

So using also the fact mentioned in the problem set, we further obtain

$$\mathbb{P}(X_n \neq Y_n) \leq C \cdot \exp(-\gamma n)$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\mathbb{P}(X_n \neq Y_n)\right) \leq \lim_{n\to\infty} \frac{1}{n}(\log C - \gamma n) = -\gamma = \frac{\sqrt{5}-3}{8} \simeq -0.095$$

Bonus (2 points). No, the process M is not a Markov chain. Here is why.

We can rewrite the definition of M_n as:

$$M_n = \begin{cases} 1, & \text{if } D_n \in \{0, 1\} \\ 2, & \text{if } D_n = 2 \end{cases}$$

Intuitively, knowing that at time n, it holds that $M_n = 1$ does not provide enough information to deduce with which probability $M_{n+1} = 1$. If it happens that $D_n = 0$, then this probability is equal to 1; if on the other hand $D_n = 1$, then this probability is equal to $3/4$.