Midterm Exam

Please pay attention to the presentation of your answers **(2 points)**.

Exercise 1. (24 points)

Let $p, q \in [0,1]$ be two numbers such that $p+q=1$ and $(X_n,\, n \geq 0)$ be a Markov chain with state space $S = \mathbb{Z} \times \{-1,+1\}$ and the following behaviour:

If $X_n = (i,u)$, then $X_{n+1} = (i+u,u)$ with probabilty p and $X_{n+1} = (i,-u)$ with probability q.

Let us also denote by P the corresponding transition matrix.

a) (3 points) Draw the transition graph of the chain $(X_n,\, n \geq 0)$.
NB: Of course, only a part of the whole graph can be drawn !

b) (3 points) Depending on the value of $0 \leq p \leq 1$, describe the equivalence classes of the chain $(X_n,\, n \geq 0)$.

From now on in this exercise, assume that $0 < p < 1$.
Moreover, for notational simplicity, states of the chain may be written as $i,+$ or $i,-$ for $i \in \mathbb{Z}$.

c) (3 points) What is the periodicity of state $0,+$? Justify your answer.

Fact. One can show that $(X_n,\, n \geq 0)$ is recurrent for every value of $0 < p < 1$.

Bonus (2 points). Provide a (short) intuitive argument why this is the case.

d) (4 points) Here is another fact : one can show that if π were a stationary distribution of the chain, then necessarily $\pi_{i,+} = \pi_{i,-}$ for all $i \in \mathbb{Z}$. Show that this would also imply that $\pi_{i,+} = \pi_{j,+} = \pi_{i,-} = \pi_{j,-}$ for all $i,j \in \mathbb{Z}$.

e) (3 points) What do you deduce from d) on the positive-recurrence/null-recurrence of the chain ?

Let now $(Y_n,\, n \geq 0)$ be the chain whose transition matrix is given by $Q = P^2$.

f) (4 points) Compute the entries of the matrix Q.

g) (4 points) Describe the equivalence classes of the chain $(Y_n,\, n \geq 0)$.

Hint : For these last two questions, it might help drawing the transition graph of $(Y_n,\, n \geq 0)$
(but this is not asked).

Exercise 2 (24 points).

Let $(X_n,\, n \geq 0)$, $(Y_n,\, n \geq 0)$ be two cyclic and symmetric random walks on the state space $S = \{0, 1, 2, 3, 4\}$ such that

- X_0 and Y_0 take different values (we suppose these to be deterministic) ;
- X and Y are coupled in the following way : they evolve independently until they meet in a common state; from there on, they evolve together.

Let also $d(i, j)$ denote the cyclic distance between two states $i, j \in S$, that is :

$$d(i, j) = \min\{|i - j|, 5 - |i - j|\}$$

so that for example, $d(2, 3) = 1$, while $d(1, 4) = 2$.

We consider in the following the stochastic process D defined as $(D_n = d(X_n, Y_n),\, n \geq 0)$.

Fact. The process D is also a Markov chain.

a) (4 points) Compute the state space of D, as well as its transition matrix P.

b) (3 points) Compute the equivalence classes of D ; which are recurrent/transient ?

c) (4 points) Does D admit a unique stationary distribution π ? If yes, compute it. Is it also a limiting distribution ? Justify.

d) (6 points) Compute the eigenvalues of P.

e) (3 points) Show that for any probability distribution μ on a state space S containing state 0, it holds that

$$\|\mu - \delta_0\|_{\mathrm{TV}} = 1 - \mu_0$$

Fact. In the spirit of what was shown in the lectures (and even if the setup is slightly different here), one can show that there exists a constant $C > 0$ such that

$$\|\pi^{(n)} - \pi\|_{\mathrm{TV}} \leq C \cdot \exp(-\gamma n), \quad \forall n \geq 1$$

where $\pi^{(n)}$ is the distribution of D_n and γ is the spectral gap of the chain.

f) (4 points) From part e) and the above fact, deduce an (interesting) upper bound on

$$\lim_{n \to \infty} \frac{1}{n} \log(\mathbb{P}(X_n \neq Y_n))$$

A numerical value is expected here.

Bonus (2 points). Consider now the stochastic process $(M_n,\, n \geq 0)$ defined as

$$M_n = \begin{cases} 1 & \text{if } d(X_n, Y_n) \leq 1 \\ 2 & \text{if } d(X_n, Y_n) > 1 \end{cases}$$

Is this process also a Markov chain ? Provide a (short) justification / intuition.