Solution Set 11

Problem 1: Properties

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, X be an integrable random variable defined on this space and let $\mathcal{G}$ be a sub-σ-field of $\mathbb{F}$. Relying only on the definition of conditional expectation, show the following properties:

a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

b) If X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

c) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

d) If Y is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})\,Y$ a.s.

e) If $\mathcal{H}$ is a sub-σ-field of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})$ a.s.

Hint for parts b) to e): According to the course definition, in order to check that some candidate random variable Z is the conditional expectation of X given $\mathcal{G}$, you should check the following two conditions:

(i) Z is $\mathcal{G}$-measurable;

(ii) Z satisfies $\mathbb{E}((Z - X)\,U) = 0$ for every U $\mathcal{G}$-measurable and bounded.

Solution a) Use part (ii) of the definition with $U \equiv 1$ (such a U belongs to G).

b) (i) $Z = \mathbb{E}(X)$ is constant and therefore $\mathcal{G}$-measurable; (ii) Let $U \in G$: $\mathbb{E}(XU) = \mathbb{E}(X)\,\mathbb{E}(U) = \mathbb{E}(\mathbb{E}(X)\,U) = \mathbb{E}(ZU)$ (using the independence of X and U and the linearity of expectation).

c) (i) $Z = X$ is $\mathcal{G}$-measurable by assumption; (ii) Let $U \in G$: $\mathbb{E}(XU) = \mathbb{E}(ZU)$!

d) (i) $Z = \mathbb{E}(X|\mathcal{G})\,Y$ is $\mathcal{G}$-measurable; (ii) Let $U \in G$: $\mathbb{E}(XYU) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})\,YU)$, because part (ii) of the definition of $\mathbb{E}(X|\mathcal{G})$ implies the previous equality (indeed, $YU \in G$). Therefore, $\mathbb{E}(XYU) = \mathbb{E}(ZU)$.

e) Let us first check the left-hand side equality: $\mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable, therefore $\mathcal{G}$-measurable, so one can apply property c).

For the right-hand side equality, one has: (i) $Z = \mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable; (ii) Let $U \in H$:

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})\,U) = \mathbb{E}(\mathbb{E}(XU|\mathcal{G})) = \mathbb{E}(XU) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})\,U) = \mathbb{E}(ZU)$$

using successively d), a) and the definition of $\mathbb{E}(X|\mathcal{H})$.

Problem 2: Discrete conditional expectation

Let X, Y be two discrete random variables (with values in a countable set C). Let us moreover assume that X is integrable.

a) Show that the random variable $\psi(Y)$, where ψ is defined as

$$\psi(y) = \sum_{x \in C} x \, \mathbb{P}(\{X = x\} | \{Y = y\})$$

matches the definition of conditional expectation $\mathbb{E}(X|Y)$ given in the lectures.

b) *Application:* One rolls two independent and balanced dice (say Y and Z), each with four faces. What is the conditional expectation of the maximum of the two, given the value of one of them?

Solution a) We must check that $\mathbb{E}(\psi(Y)\, g(Y)) = \mathbb{E}(X\, g(Y))$ for any continuous and bounded function g. The computation gives indeed:

$$\mathbb{E}(\psi(Y)\, g(Y)) = \sum_{y \in C} \psi(y)\; g(y)\; \mathbb{P}(\{Y = y\}) = \sum_{x,y \in C} x\; g(y)\; \mathbb{P}(\{X = x, Y = y\}) = \mathbb{E}(X\, g(Y))$$

b) Let Y and Z be the two independent dice rolls: $\mathbb{P}(\{Y = i\}) = \mathbb{P}(\{Z = j\}) = 0.25$ and $\mathbb{P}(\{Y = i, Z = j\}) = \mathbb{P}(\{Y = i\})\, \mathbb{P}(\{Z = j\})$. We therefore have $\mathbb{E}(\max(Y,Z)|Y) = \psi(Y)$, where

$$\begin{aligned}
\psi(i) &= \sum_{j=i}^{4} \max(i,j)\, \mathbb{P}(\{\max(Y,Z) = j\}|\{Y = i\}) = \sum_{j=i}^{4} \max(i,j)\, \frac{\mathbb{P}(\{\max(Y,Z) = j, Y = i\})}{\mathbb{P}(\{Y = i\})} \\
&= i\, \frac{\mathbb{P}(\{Z \le i, Y = i\})}{\mathbb{P}(\{Y = i\})} + \sum_{j=i+1}^{4} j\, \frac{\mathbb{P}(\{Z = j, Y = i\})}{\mathbb{P}(\{Y = i\})} = i\, \mathbb{P}(\{Z \le i\}) + \sum_{j=i+1}^{4} j\, \mathbb{P}(\{Z = j\})
\end{aligned}$$

So $\psi(1) = 2.5$, $\psi(2) = 2.75$, $\psi(3) = 3.25$ and $\psi(4) = 4$.

Problem 3: Zombie apocalypse

A zombie bites N people during its lifetime (un-death time?), where N has the Poisson distribution with parameter λ. Once a person is bitten, they become a zombie with probability p or their brain is eaten with probability $q = 1 - p$, independently of all other bitten individuals. Let Z be the number of zombie offsprings created by a zombie.

a) Find $\mathbb{E}(Z|N)$ and $\mathbb{E}(Z)$

b) Find $\mathbb{E}(N|Z)$

c) *Optional*: Assume that the initial zombie infestation is small (relative to the population of the planet). According to the above model, for which values of p and λ would you expect the infestation to turn into a full blown apocalypse and for which values would it die down locally?

Solution a) From the previous Exercise,

$$\psi(n) = \sum_{k \ge 0} k \mathbb{P}(\{Z = k\}|\{N = n\}) = \sum_{k \ge 1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = pn$$

and so $\mathbb{E}(Z|N) = pN$. We also have that

$$\mathbb{E}(Z) = \mathbb{E}(\mathbb{E}(Z|N)) = \mathbb{E}(pN) = p\lambda.$$

b) We have that

$$\mathbb{P}(\{N = k\}|\{Z = z\}) = \frac{\mathbb{P}(\{N = k, Z = z\})}{\{Z = z\})} = \frac{\binom{k}{z} p^z q^{k-z} (\lambda^k/k!) e^{-\lambda}}{\sum_{m \ge z} \binom{m}{z} p^z q^{m-z} (\lambda^m/m!) e^{-\lambda}} = \frac{(q\lambda)^{k-z}}{(k-z)!} e^{-q\lambda}$$

From the previous Exercise,

$$\psi(z) = \sum_{k\geq 0} k\mathbb{P}(\{N = k\}|\{Z = z\}) = \sum_{k\geq z}^{n} k\frac{(q\lambda)^{k-z}}{(k-z)!}e^{-q\lambda} = z + q\lambda$$

and so $\mathbb{E}(N|Z) = Z + q\lambda$.

c) Intuitively, we would like each zombie to produce less than one offspring, so the infestation will die down if $p\lambda < 1$. However, even if this condition is met, we will also need λ to be small (relative to the population of the planet). Otherwise, we risk having a few very hungry zombies eating the brains of some significant percentage of world's population before disappearing themselves.