Solution to Additional exercises to test yourself
Introduction to Quantum Information Processing

Exercise 1 *Entropy of reduced states for mixed global states*

a) The equality

$$S(\rho_A) = S(\rho_B)$$

holds only when the global state ρ_{AB} is *pure*. If ρ_{AB} is mixed, this equality generally *does not hold*. In other words, for a mixed state, the reduced density matrices $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ and $\rho_B = \mathrm{Tr}_A(\rho_{AB})$ may have different von Neumann entropies.

b) Consider a 2-qubit system with the following mixed state:

$$\rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |01\rangle\langle 01|\right).$$

Compute the reduced density matrices:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 0|\right) = |0\rangle\langle 0|,$$

$$\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}I_2.$$

Now compute the von Neumann entropies:

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A) = -\mathrm{Tr}(|0\rangle\langle 0| \log |0\rangle\langle 0|) = 0,$$

$$S(\rho_B) = -\mathrm{Tr}\left(\frac{1}{2}I_2 \log \frac{1}{2}I_2\right) = \log 2 = 1.$$

Therefore, we have an explicit example of a mixed state ρ_{AB} where

$$S(\rho_A) \neq S(\rho_B).$$

Exercise 2 *Basis changes*

a) The Hadamard basis (also called the $|+\rangle, |-\rangle$ basis) is defined as

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

b) The 2-qubit Hadamard basis is obtained by taking tensor products of the single-qubit Hadamard basis:

$$\{|+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|-\rangle\}.$$

Using the hint, for any orthonormal basis $\{|\phi_i\rangle\}$:

$$|\psi\rangle = \sum_i \langle\phi_i|\psi\rangle\, |\phi_i\rangle.$$

Denote the 2-qubit Hadamard basis as:

$$|\phi_1\rangle = |+\rangle|+\rangle, \quad |\phi_2\rangle = |+\rangle|-\rangle, \quad |\phi_3\rangle = |-\rangle|+\rangle, \quad |\phi_4\rangle = |-\rangle|-\rangle.$$

Thanks to the hint, the expansion of $|\psi\rangle$ in this basis is:

$$|\psi\rangle = \sum_{i=1}^{4} \langle\phi_i|\psi\rangle\, |\phi_i\rangle.$$

First, let's write the Hadamard states in the computational basis:

$$|+\rangle|+\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle),$$

$$|+\rangle|-\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle),$$

$$|-\rangle|+\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle),$$

$$|-\rangle|-\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle).$$

Then the coefficients are

$$c_1 = \langle\phi_1|\psi\rangle = \frac{1}{2}(x_1 + x_2 + x_3 + x_4),$$

$$c_2 = \langle\phi_2|\psi\rangle = \frac{1}{2}(x_1 - x_2 + x_3 - x_4),$$

$$c_3 = \langle\phi_3|\psi\rangle = \frac{1}{2}(x_1 + x_2 - x_3 - x_4),$$

$$c_4 = \langle\phi_4|\psi\rangle = \frac{1}{2}(x_1 - x_2 - x_3 + x_4).$$

giving the Hadamard-basis expansion:

$$|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + c_3|\phi_3\rangle + c_4|\phi_4\rangle.$$

c) (Bonus) Let $\{|\phi_i\rangle\}_{i=1}^m$ be an orthonormal basis of an m-dimensional Hilbert space $\mathcal{H}$. Then any $|\psi\rangle \in \mathcal{H}$ can be decomposed in this basis as

$$|\psi\rangle = \sum_{i=1}^{m} \alpha_i |\phi_i\rangle$$

for some coefficients $\alpha_i \in \mathbb{C}$.

Using the orthonormality of the basis, $\langle\phi_j|\phi_i\rangle = \delta_{ij}$, we compute

$$\langle\phi_j|\psi\rangle = \sum_{i=1}^{m} \alpha_i \langle\phi_j|\phi_i\rangle = \sum_{i=1}^{m} \alpha_i \delta_{ji} = \alpha_j.$$

Therefore, the coefficients are

$$\alpha_i = \langle\phi_i|\psi\rangle,$$

which gives

$$|\psi\rangle = \sum_{i=1}^{m} \langle\phi_i|\psi\rangle\, |\phi_i\rangle.$$

Exercise 3 *Criterion for pure states*

a) $\Rightarrow:$ If ρ is pure, then $\rho = |\psi\rangle\langle\psi|$ for some state $|\psi\rangle$. Then

$$\rho^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi| = \rho,$$

so $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$.

$\Leftarrow:$ if $\mathrm{Tr}(\rho^2) = 1$, write ρ in its eigenbasis $\rho = \sum_i \lambda_i |i\rangle\langle i|$ where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$ since ρ is hermitian and semi positive definite. Then

$$\mathrm{Tr}\big(\rho^2\big) = \sum_i \lambda_i^2 = 1.$$

Since $\sum_i \lambda_i = 1$, the only way $\sum_i \lambda_i^2 = 1$ is if one eigenvalue is 1 and all others are 0. Hence $\rho = |i\rangle\langle i|$ is pure.

b) If the global state ρ_{AB} is pure, then the purities of the reduced states are equal: $\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_B^2)$. Any pure state $|\psi\rangle_{AB}$ admits a Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |i\rangle_A \otimes |i\rangle_B,$$

with $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$. Then

$$\rho_A = \sum_i \lambda_i |i\rangle\langle i|_A, \qquad \rho_B = \sum_i \lambda_i |i\rangle\langle i|_B.$$

Hence, ρ_A and ρ_B have the same eigenvalues $\{\lambda_i\}$, and therefore $\mathrm{Tr}(\rho_A^2) = \sum_i \lambda_i^2 = \mathrm{Tr}(\rho_B^2)$.

Example where this is not the case: consider the mixed state

$$\rho_{AB} = \frac{1}{2}\left|00\right\rangle\!\left\langle 00\right| + \frac{1}{2}\left|01\right\rangle\!\left\langle 01\right| = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \left|0\right\rangle\left\langle 0\right| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_B = \mathrm{Tr}_A(\rho_{AB}) = \frac{1}{2}I = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Compute the purities:

$$\mathrm{Tr}\big(\rho_A^2\big) = 1, \qquad \mathrm{Tr}\big(\rho_B^2\big) = \frac{1}{2}.$$

Hence, $\mathrm{Tr}(\rho_A^2) \neq \mathrm{Tr}(\rho_B^2)$.

Exercise 4 *Measurement*

a) Consider the two-qubit state

$$\left|\psi\right\rangle = \frac{\left|00\right\rangle - \left|11\right\rangle}{\sqrt{2}}.$$

We want to compute the expectation value of the observable $X \otimes Z$, where X acts on the first qubit and Z on the second. The expectation value is

$$\langle XZ\rangle = \left\langle\psi\right| X \otimes Z \left|\psi\right\rangle.$$

First, compute $X \otimes Z \left|\psi\right\rangle$:

$$X \otimes Z \left|\psi\right\rangle = \frac{1}{\sqrt{2}}((X \otimes Z)\left|00\right\rangle - (X \otimes Z)\left|11\right\rangle = \frac{1}{\sqrt{2}}(\left|10\right\rangle + \left|01\right\rangle).$$

Then,

$$\langle X \otimes Z\rangle = \frac{1}{2}(\left\langle 00\right| - \left\langle 11\right|)(\left|10\right\rangle + \left|01\right\rangle) = \frac{1}{2}\left(\langle 00|10\rangle - \langle 11|10\rangle + \langle 00|01\rangle - \langle 11|01\rangle\right) = 0.$$

b) Now consider the mixed state

$$\rho = p\ \left|\psi\right\rangle\!\left\langle\psi\right| + (1-p)\left|0\right\rangle\!\left\langle 0\right| \otimes \left|+\right\rangle\!\left\langle +\right|$$

For $\left|\psi\right\rangle = \frac{\left|00\right\rangle - \left|11\right\rangle}{\sqrt{2}}$ and $\left|0+\right\rangle = \left|0\right\rangle \otimes \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}$, the matrix forms are:

$$\left|\phi\right\rangle\!\left\langle\phi\right| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad \left|0+\right\rangle\!\left\langle 0+\right| = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the density matrix ρ is

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1-p & 0 & -p \\ 1-p & 1-p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & 0 & p \end{pmatrix}.$$

The expectation value is

$$\langle X \otimes Z \rangle_\rho = \mathrm{Tr}(\rho\,(X \otimes Z)) = p\,\underbrace{\mathrm{Tr}(|\phi\rangle\langle\phi|\,(X \otimes Z))}_{=0 \text{ by a)}} + (1-p)\,\mathrm{Tr}(|0\rangle\langle 0| \otimes |+\rangle\langle +|\,(X \otimes Z)).$$

For the second term, we have

$$\mathrm{Tr}(|0\rangle\langle 0| \otimes |+\rangle\langle +|\,(X \otimes Z)) = \mathrm{Tr}(|0\rangle\langle 1| \otimes |+\rangle\langle -|) = \mathrm{Tr}(|0\rangle\langle 1|)\,\mathrm{Tr}(|+\rangle\langle -|) = 0.$$

Thus for all p, the expectation value is zero.

Exercise 5 *Hermitian Operators and Tensor Products*

Let A and B be Hermitian operators acting on 2-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

a) We can look at the coefficients of the matrix $A \otimes B$ in the computational basis. The matrix elements are given by $\langle ij|\,A \otimes B\,|kl\rangle = \langle i|\,A\,|k\rangle\,\langle j|\,B\,|l\rangle$. Taking the Hermitian conjugate, we have

$$\begin{aligned}(\langle ij|\,(A \otimes B)^\dagger\,|kl\rangle) &= (\langle kl|\,(A \otimes B)\,|ij\rangle)^* = (\langle k|\,A\,|i\rangle\,\langle l|\,B\,|j\rangle)^* = (\langle k|\,A\,|i\rangle)^*(\langle l|\,B\,|j\rangle)^* \\ &= (\langle i|\,A^\dagger\,|k\rangle)(\langle j|\,B^\dagger\,|l\rangle) = \langle ij|\,(A^\dagger \otimes B^\dagger)\,|kl\rangle\,.\end{aligned}$$

Since A and B are Hermitian, $A^\dagger = A$ and $B^\dagger = B$. Therefore,

$$\langle ij|\,(A \otimes B)^\dagger\,|kl\rangle = \langle ij|\,A \otimes B\,|kl\rangle\,,$$

and all matrix elements of $(A \otimes B)^\dagger$ equal those of $A \otimes B$. Hence, $A \otimes B$ is Hermitian.

b) Suppose $|\psi\rangle \in \mathcal{H}_1$ is an eigenvector of A with eigenvalue λ, and $|\phi\rangle \in \mathcal{H}_2$ is an eigenvector of B with eigenvalue μ. Then

$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = (A|\psi\rangle) \otimes (B|\phi\rangle) = (\lambda|\psi\rangle) \otimes (\mu|\phi\rangle) = (\lambda\mu)(|\psi\rangle \otimes |\phi\rangle).$$

Hence, $|\psi\rangle \otimes |\phi\rangle$ is an eigenvector of $A \otimes B$ with eigenvalue $\lambda\mu$.

The tensor product $A \otimes B$ has $dim(\mathcal{H}_1) \times dim(\mathcal{H}_2)$ eigenvalues in total.

c) Let

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Check Hermiticity:

$$A^\dagger = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = A, \quad B^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B.$$

So both A and B are Hermitian.

Compute $A \otimes B$:

$$A \otimes B = \begin{pmatrix} 1 \cdot B & i \cdot B \\ (-i) \cdot B & 1 \cdot B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & 1 & 0 \end{pmatrix}.$$

Check Hermiticity of $A \otimes B$:

$$(A \otimes B)^\dagger = \begin{pmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & 1 & 0 \end{pmatrix} = A \otimes B,$$

which confirms that $A \otimes B$ is Hermitian.

d) The eigenvalues of A are $\{0, 2\}$ and those of B are $\{-1, 1\}$. The eigenvalues of $A \otimes B$ are the products of the eigenvalues of A and B : $\{-2, 0, 0, 2\}$.

Exercise 6 *Partial Measurement*

Consider the three-qubit state

$$|\Psi\rangle = \frac{1}{2}\Big(|001\rangle + e^{i\pi/4} |011\rangle - |100\rangle + |010\rangle \Big).$$

We measure only the second qubit (middle qubit) in the computational basis $\{|0\rangle, |1\rangle\}$.

a) Define the projectors:

$$P_0 = I \otimes |0\rangle \langle 0| \otimes I, \quad P_1 = I \otimes |1\rangle \langle 1| \otimes I.$$

The probability to measure qubit 2 in state $|0\rangle$ is

$$p(0) = \langle \Psi | P_0 | \Psi \rangle .$$

Apply P_0 to $|\Psi\rangle$: it keeps only terms where the second qubit is 0:

$$P_0 |\Psi\rangle = \frac{1}{2}(|001\rangle - |100\rangle).$$

Thus,

$$p(0) = \langle \Psi | P_0 | \Psi \rangle = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Similarly, the probability to measure qubit 2 in state $|1\rangle$ is

$$P_1 |\Psi\rangle = \frac{1}{2}(e^{i\pi/4} |011\rangle + |010\rangle),$$

$$p(1) = \langle\Psi| P_1 |\Psi\rangle = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Verification:

$$p(0) + p(1) = \frac{1}{2} + \frac{1}{2} = 1.$$

b) After measuring qubit 2 with outcome 0, the state collapses to

$$|\Psi_0\rangle = \frac{P_0 |\Psi\rangle}{\sqrt{p(0)}} = \frac{\frac{1}{2}(|001\rangle - |100\rangle)}{\sqrt{1/2}} = \frac{1}{\sqrt{2}}(|001\rangle - |100\rangle).$$

After measuring qubit 2 with outcome 1, the state collapses to

$$|\Psi_1\rangle = \frac{P_1 |\Psi\rangle}{\sqrt{p(1)}} = \frac{\frac{1}{2}(|010\rangle + e^{i\pi/4} |011\rangle)}{\sqrt{1/2}} = \frac{1}{\sqrt{2}}(|010\rangle + e^{i\pi/4} |011\rangle).$$

c)
- For $|\Psi_0\rangle$:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|001\rangle - |100\rangle) = \frac{1}{\sqrt{2}}(|0\rangle_1 |0\rangle_2 |1\rangle_3 - |1\rangle_1 |0\rangle_2 |0\rangle_3) = \frac{1}{\sqrt{2}}(|0\rangle_1 |1\rangle_3 - |1\rangle_1 |0\rangle_3) \otimes |0\rangle_2 .$$

  The state of qubits 1 and 3 is a Bell state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, which is entangled.

- For $|\Psi_1\rangle$:

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|010\rangle + e^{i\pi/4} |011\rangle) = |0\rangle_1 \otimes |1\rangle_2 \otimes \frac{1}{\sqrt{2}}(|0\rangle_3 + e^{i\pi/4} |1\rangle_3).$$

  The state of qubits 1 and 3 is separable: $|0\rangle_1 \otimes \frac{1}{\sqrt{2}}(|0\rangle_3 + e^{i\pi/4} |1\rangle_3)$.