Exercise 1 *Errors in the Bloch ball*

a) A bit-flip error can be seen as the environment somehow applying a X-gate to the qubit, transforming our state to

$$\rho_{b-flip} = X\rho X = \frac{1}{2}(\mathbf{I} + \vec{r}_{b-flip} \cdot \vec{\sigma}), \tag{1}$$

where we defined the bit-flipped Bloch vector as

$$\begin{aligned} \vec{r}_{b-flip} &= X(\vec{r} \cdot \sigma)X &(2)\\ &= X(r_x X + r_y Y + r_z Z)X &(3)\\ &= X(r_x \mathbf{I} - i r_y Z + i r_z Y) &(4)\\ &= r_x X - r_y Y - r_z Z. &(5) \end{aligned}$$

We immediately see that, geometrically, a bit-flip is equivalent to two rotations of 180° about the y- and the z-axis, respectively. Now, if such an event happens with probability p, then state is described by the following convex combination,

$$\rho' = (1-p)\rho + pX\rho X = \frac{1}{2}\left[\mathbf{I} + \big(r_x, (1-2p)r_y, (1-2p)r_z\big) \cdot \vec{\sigma}\right]. \tag{6}$$

We see that the new Bloch vector is defined as $\vec{r'} = \big(r_x, (1-2p)r_y, (1-2p)r_z\big)$ and observe that the x-axis remains unaffected.

(a) Before bit-flip (b) After bit-flip

Figure 1: Bit-flip error on the Bloch sphere with probability $p = 0.4$. We can see that the x-component of the Bloch vector remains unchanged, while the y- and z-components are scaled by a factor of $(1 - 2p)$.

b) A similar analysis can be done for a phase-flip error, which is described by,

$$\rho_{p-flip} = Z\rho Z = \frac{1}{2}(\mathbf{I} + \vec{r}_{p-flip} \cdot \vec{\sigma}), \tag{7}$$

where we defined the phase-flip error Bloch vector as,

$$\vec{r}_{p-flip} = -r_x X - r_y Y + r_z Z. \tag{8}$$

So, on the Bloch sphere, a phase-flip error corresponds to two rotations of 180° about the x- and y-axis, respectively. When a phase flip happens with probability p, the state is therefore described by,

$$\rho'' = (1-p)\rho + pZ\rho Z = \frac{1}{2}\left[\mathbf{I} + \big((1-2q)r_x, (1-2q)r_y, r_z\big) \cdot \vec{\sigma}\right], \tag{9}$$

with the new Bloch vector $\vec{r''} = \big((1-2q)r_x, (1-2q)r_y, r_z\big)$

(a) Before phase-flip (b) After phase-flip

Figure 2: Phase-flip error on the Bloch sphere with probability $p = 0.4$. We can see that the x-component of the Bloch vector remains unchanged, while the y- and z-components are scaled by a factor of $(1-2p)$.

c) We compute the state obtained after applying the channel, generalizing what we have seen in a) and b), i.e. that applying a Pauli P leaves the component of the Bloch vector in that direction unchanged, while doing a rotation of 180° in the two other directions. So,

$$\mathcal{E}(\rho) = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4} \tag{10}$$

$$= \frac{1}{4}\Bigg(\frac{\mathbf{I} + (r_x, r_y, r_z)\cdot\vec{\sigma}}{2} + \frac{\mathbf{I} + (r_x, -r_y, r_z)\cdot\vec{\sigma}}{2} + \tag{11}$$

$$+ \frac{\mathbf{I} + (-r_x, r_y, -r_z)\cdot\vec{\sigma}}{2} + \frac{\mathbf{I} + (-r_x, -r_y, r_z)\cdot\vec{\sigma}}{2}\Bigg)$$

$$= \frac{\mathbf{I}}{2}. \tag{12}$$

This channel maps any 1-qubit state to the fully-mixed state $\mathbf{I}/2$, i.e. to the center of the Bloch sphere.

d) A depolarizing channel with probability p is expressed as

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z).$$

We proceed similarly as in c), to express the new Bloch vector $\vec{r'}$:

$$\begin{aligned}
\mathcal{E}(\rho) &= (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z) && (13)\\
&= (1-p)\frac{\mathbf{I} + (r_x, r_y, r_z)\cdot\vec{\sigma}}{2} + \frac{p}{3}\Bigg(\frac{\mathbf{I} + (r_x, -r_y, -r_z)\cdot\vec{\sigma}}{2} + && (14)\\
&\qquad + \frac{\mathbf{I} + (-r_x, r_y, -r_z)\cdot\vec{\sigma}}{2} + \frac{\mathbf{I} + (-r_x, -r_y, r_z)\cdot\vec{\sigma}}{2}\Bigg) \\
&= \frac{1}{2}\left[\mathbf{I} + \left((1-p) - \frac{p}{3}\right)(r_x, r_y, r_z)\cdot\vec{\sigma}\right] && (15)\\
&= \frac{1}{2}\left[\mathbf{I} + \left(1 - \frac{4p}{3}\right)\vec{r}\cdot\vec{\sigma}\right] && (16)\\
&= \frac{1}{2}\left[\mathbf{I} + \vec{r'}\cdot\vec{\sigma}\right], && (17)
\end{aligned}$$

where we defined the new Bloch vector $\vec{r'} = (1 - \frac{4p}{3})\vec{r}$.

(a) Before (b) After depolarising noise

Figure 3: Depolarising noise on the Bloch sphere with probability $p = 0.4$. All components of the Bloch vector are scaled by the same factor.

e) We decompose the situation in two sequences. First, we have the probability of having a bit-flip with probability p_X, which affects our state in the following way,

$$\rho_1 = \rho' = \frac{1}{2}\left[\mathbf{I} + \vec{r'}\cdot\vec{\sigma}\right], \qquad (18)$$

where we reused the result in a), i.e. $\vec{r'} = \left(r'_x, r'_y, r'_z\right) = \left(r_x, (1-2p_X)r_y, (1-2p_X)r_z\right)$. Second, the qubit has probability of undergoing a phase-flip with probability p_Z; we use the result from b) but using the components r'_x, r'_y and r'_z instead of r_x, r_y and r_z.

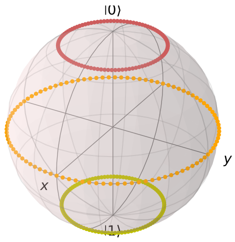
$$\begin{aligned}
\rho_2 &= \frac{1}{2}\left[\mathbf{I} + \left((1-2p_Z)r'_x, (1-2p_Z)r'_y, r'_z\right)\cdot\vec{\sigma}\right] \\
&= \frac{1}{2}\left[\mathbf{I} + \left((1-2p_Z)r_x, (1-2p_Z)(1-2p_X)r_y, (1-2p_X)r_z\right)\cdot\vec{\sigma}\right]
\end{aligned} \qquad (19)$$

We have seen that in the case of a depolarising noise, in question d), each component of $\vec{r}$ is scaled by the same factor. Thus, we need

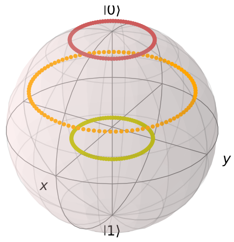
$$1 - 2p_X = 1 - 2p_Z = (1 - 2p_X)(1 - 2p_Z). \tag{20}$$

This is true only for $p_X = p_Z = 0$ or $\frac{1}{2}$. So there are either no errors or we are in the case of question c).

f) The Bloch vector after amplitude damping with probability p are rescaled and shifted toward $|0\rangle$.



(a) Before



(b) After amplitude damping

Figure 4: Amplitude damping noise on the Bloch sphere with $\gamma = 0.4$.

Exercise 2 *Three-qubit repetition code*

a) If phase-flip error happens with one of physical qubits:

$$Z_1|0\rangle_L = Z_1|000\rangle = |000\rangle = |0\rangle_L$$
$$Z_1|1\rangle_L = Z_1|111\rangle = -|111\rangle = -|1\rangle_L$$

Similarly, if an error happens with second and third physical qubit. Since a phase-flip add only a global phase, it does not add a logical error on the logical states.

For a logical qubit $|\phi\rangle = \alpha|0\rangle_L + \beta|1\rangle_L$:

$$Z_1|\phi\rangle = Z_1\left(\alpha|000\rangle + \beta|111\rangle\right) = \alpha|0\rangle_L - \beta|1\rangle_L \tag{21}$$

Therefore, physical phase flip corresponds to a relative phase flip in a logical qubit.

However, if phase-flip hapens with 2 physical qubits, it doesn't induce any logical error. Indeed,

$$Z_1 \otimes Z_2 \otimes I|0\rangle_L = Z_1 \otimes Z_2 \otimes I|000\rangle = |0\rangle_L$$
$$Z_1 \otimes Z_2 \otimes I|1\rangle_L = Z_1 \otimes Z_2 \otimes I|111\rangle = (-1)^2|1\rangle_L$$

b) If a Y error happens with one of physical qubits:

$$Y_1|0\rangle_L = Y_1|000\rangle = -i|100\rangle = X_1|0\rangle_L$$
$$Y_1|1\rangle_L = Y_1|111\rangle = i|011\rangle = X_1|1\rangle_L$$

Similarly, if an error happens with second and third physical qubit. Up to a global phase, a Y error is equivalent to a bit flip error X for a logical state.

For a logical qubit, $|\phi\rangle = \alpha|0\rangle_L + \beta|1\rangle_L$:

$$Y_1|\phi\rangle = Y_1(\alpha|000\rangle + \beta|111\rangle) = i(-\alpha|100\rangle + \beta|011\rangle) \tag{22}$$

Therefore, a physical Y error corresponds to a relative phase flip and a bit flip in a logical qubit.

c) For depolarizing channel with error probability p, the probabilities of bit flip (X), phase flip (Z) and Y are $\frac{p}{3}$. Since a logical state is not sensible to phase-flip errors and Y errors are equivalent to bit-flip errors, we can consider only bit-flip errors in the analysis. The effective physical bit-flip probability per qubit is then:

$$p_X = \frac{2p}{3}.$$

d)
- First case: the logical state is unaffected.

  The logical state is unaffected if no error occurs. Probability that no error happens in all 3 physical qubits:

  $$P_{\text{no error}} = (1 - p_X)^3.$$

  There are also the situations when logical state is unaffected but physical changes. It happens because logical state remain unaffected when majority voting doesn't fail. For example, if one physical bit flip happens $|000\rangle \to |100\rangle$, it will still correspond to $|0\rangle_L \to |0\rangle_L$. The majority voting doesn't fail if only one physical bit flip happens. Probability of one bit flip:

  $$P_{1 \text{ bit flip}} = 3p_X(1 - p_X)^2$$

  Therefore the probability that logical state is unaffected is:

  $$P_{\text{unaffected}} = (1-p_X)^3 + 3p_X(1-p_X)^2 = \left(1 - \frac{2p}{3}\right)^3 + 2p\left(1 - \frac{2p}{3}\right)^2 = 1 - \frac{4}{3}p^2 + \frac{16}{27}p^3.$$

- Logical bit flip occurs when majority voting fails:

  $$\text{if the original state is } |0\rangle_L \text{ and errors creates } \left.\begin{array}{l}|011\rangle\\|101\rangle\\|110\rangle\\|111\rangle\end{array}\right\} \text{the majority will votes } |1\rangle_L$$

  $$\text{if the original state is } |1\rangle_L \text{ and errors creates } \left.\begin{array}{l}|001\rangle\\|010\rangle\\|100\rangle\\|000\rangle\end{array}\right\} \text{the majority will votes } |0\rangle_L$$

  Thus, to have logical bit-flip we need physical bit-flip on two or three qubits.

In total,

$$\begin{aligned} P_{\text{logical X}} &= P_{\text{2 bit-flips}} + P_{\text{3 bit-flips}} \\ &= 3\left(p_X\right)^2\left(1-p_X\right)+\left(p_X\right)^3 \\ &= 3\left(\frac{2p}{3}\right)^2\left(1-\frac{2p}{3}\right)+\left(\frac{2p}{3}\right)^3 \\ &= \frac{4}{3}p^2-\frac{16}{27}p^3. \end{aligned}$$

e) Probability of failure:

$$P_{\text{fail}} = 1 - P_{\text{unaffected}} = P_{\text{Logical X}} = \frac{4}{3}p^2 - \frac{16}{27}p^3. \tag{23}$$

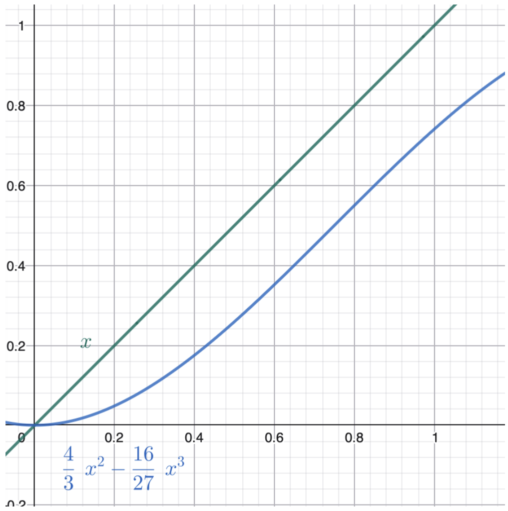


Figure 5: Probability of Error Comparison

We see that the probability p_L is always less than p, i.e. the logical error rate is lower than the physical error rate. This shows that the three-qubit repetition code is able to reduce the error rate for logical states. However, for general logical qubits, as seen in question a) and b), this code can only correct bit-flip errors, and not phase-flip errors.