Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 13 – Solutions

Exercise 1: Let $F\colon M \to N$ be a smooth map. Prove the following assertions:

(a) $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) It holds that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart $\big(V, (y^i)\big)$ on N, we have

$$F^*\left(\sum_I{}' \omega_I \, dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F) \, d\left(y^{i_1} \circ F\right) \wedge \ldots \wedge d\left(y^{i_k} \circ F\right).$$

Solution:

(a) Let $\omega, \eta \in \Omega^k(N)$ and $\lambda, \mu \in \mathbb{R}$. Fix $p \in M$ and let $v_1, \ldots, v_k \in T_pM$. We have

$$\begin{aligned}
\big(F^*(\lambda\omega + \mu\eta)\big)_p(v_1, \ldots, v_k) &= (\lambda\omega + \mu\eta)_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) \\
&= \lambda\, \omega_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) + \mu\, \eta_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) \\
&= \lambda(F^*\omega)_p(v_1, \ldots, v_k) + \mu(F^*\eta)_p(v_1, \ldots, v_k) \\
&= \big(\lambda(F^*\omega)_p + \mu(F^*\eta)_p\big)(v_1, \ldots, v_k),
\end{aligned}$$

which implies that

$$\big(F^*(\lambda\omega + \mu\eta)\big)_p = \lambda(F^*\omega)_p + \mu(F^*\eta)_p.$$

Hence, $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) Let $\omega \in \Omega^k(N)$ and $\eta \in \Omega^\ell(N)$ be arbitrary. Fix $p \in M$ and let $v_1, \ldots, v_{k+\ell} \in T_pM$. We have

$$\begin{aligned}
F^*(\omega \wedge \eta)_p\,(v_1, \ldots, v_{k+\ell}) &= (\omega \wedge \eta)_{F(p)}\big(dF_p(v_1), \ldots, dF_p(v_{k+\ell})\big) \\
&= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn}\sigma)\, \omega\left(dF_p\big(v_{\sigma(1)}\big), \ldots, dF_p\big(v_{\sigma(k)}\big)\right) \eta\left(dF_p\big(v_{\sigma(k+1)}\big), \ldots, dF_p\big(v_{\sigma(k+\ell)}\big)\right)
\end{aligned}$$

and

$$[(F^*\omega)\wedge(F^*\eta)]_p\,(v_1,\dots,v_{k+\ell}) =$$

$$= \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}} (\operatorname{sgn}\sigma)\,(F^*\omega)_p\left(v_{\sigma(1)},\dots,v_{\sigma(k)}\right)(F^*\eta)_p\left(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}\right)$$

$$= \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}} (\operatorname{sgn}\sigma)\,\omega\left(dF_p\big(v_{\sigma(1)}\big),\dots,dF_p\big(v_{\sigma(k)}\big)\right)\eta\left(dF_p\big(v_{\sigma(k+1)}\big),\dots,dF_p\big(v_{\sigma(k+\ell)}\big)\right).$$

As the above two expressions agree, we conclude that $F^*(\omega\wedge\eta) = (F^*\omega)\wedge(F^*\eta)$, as desired.

(c) The assertion follows immediately from (a), (b) and *Proposition 8.14*.

Exercise 2 (to be submitted):

(a) Let V be a finite-dimensional real vector space and let $\omega^1,\dots,\omega^k\in V^*$. Show that the covectors $\omega^1,\dots,\omega^k$ are linearly dependent if and only if $\omega^1\wedge\ldots\wedge\omega^k = 0$.

(b) Consider the smooth map

$$F\colon \mathbb{R}^2\to\mathbb{R}^2,\ (s,t)\mapsto (st, e^t)$$

and the smooth 1-forms

$$\omega = x\,dy - y\,dx,\ \ \eta = x\,dy \in \Omega^1(\mathbb{R}^2).$$

(Observe that we denote the standard coordinates on the source by (s,t) and the standard coordinates on the target by (x,y).)

(i) Compute $d\eta$ and $F^*\eta$.

(ii) Verify by direct computation that $d(F^*\eta) = F^*(d\eta)$.

(iii) Compute $\omega\wedge\eta$ and $F^*(\omega\wedge\eta)$.

(c) Consider the smooth manifolds

$$M = \left\{(u,v)\in\mathbb{R}^2 \mid u^2+v^2<1\right\} \qquad \text{and} \qquad N = \mathbb{R}^3\setminus\{0\},$$

the smooth map

$$F\colon M\to N,\ (u,v)\mapsto \left(u, v, \sqrt{1-u^2-v^2}\right),$$

and the differential forms

$$\omega = \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2(N)$$

and

$$\eta = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{1/2}} \in \Omega^1(N).$$

(i) Compute $d\omega$ and $d\eta$.

(ii) Compute $\omega \wedge \eta$ and $\eta \wedge d\eta$.

(iii) Compute $F^*\omega$ and $F^*(d\eta)$.

(iv) Verify that $d(F^*\omega) = F^*(d\omega)$.

Solution:

(a) Assume first that the covectors $\omega^1, \ldots, \omega^k$ are linearly dependent. Then there exist $j \in \{1, \ldots, k\}$ and $\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_k \in \mathbb{R}$ such that $\omega^j = \sum_{i \neq j} \lambda_i \, \omega^i$. Therefore,

$$\begin{aligned} \omega^1 \wedge \ldots \wedge \omega^{j-1} \wedge \omega^j \wedge \omega^{j+1} \wedge \ldots \wedge \omega^k &= \omega^1 \wedge \ldots \wedge \omega^{j-1} \wedge \sum_{i \neq j} \lambda_i \, \omega^i \wedge \omega^{j+1} \wedge \ldots \wedge \omega^k \\ &= \sum_{i \neq j} \lambda_i \left(\omega^1 \wedge \ldots \wedge \omega^{j-1} \wedge \omega^i \wedge \omega^{j+1} \wedge \ldots \wedge \omega^k \right) \\ &= 0 \end{aligned}$$

by [*Multilinear Algebra, Proposition C.25*(d)].

Assume now that the covectors $\omega^1, \ldots, \omega^k$ are linearly independent. We will show below that (the alternating k-multilinear function) $\eta := \omega^1 \wedge \ldots \wedge \omega^k \neq 0$. It suffices to find $v_1, \ldots, v_k \in V$ such that $\eta(v_1, \ldots, v_k) \neq 0$. To this end, set $n = \dim_{\mathbb{R}} V$ and note that $n \geq k$. Since $\omega^1, \ldots, \omega^k$ are linearly independent elements of V^*, we can complete them to a basis $\{\omega^1, \ldots, \omega^k, \omega^{k+1}, \ldots, \omega^n\}$ of V^*, and consider subsequently the basis $\{v_1, \ldots, v_n\}$ of V dual to $\{\omega^j\}$; see (the second paragraph after) [*Multilinear Algebra, Proposition C.5*]. By [*Multilinear Algebra, Proposition C.25*(d)] we then obtain

$$\eta(v_1, \ldots, v_k) = \det\Big(\big(\omega^j(v_i)\big) \Big) = \det\left(\delta_i^j\right) = 1,$$

and thus $\eta \neq 0$, as desired.

(b)(i) We have

$$d\eta = dx \wedge dy$$

and

$$F^*\eta = (st)\, d(e^t) = ste^t \, dt.$$

(b)(ii) We compute

$$d(F^*\eta) = d(ste^t) \wedge dt = \big(te^t \, ds + s(1+t)e^t \, dt\big) \wedge dt = te^t \, ds \wedge dt$$

and

$$F^*(d\eta) = d(st) \wedge d(e^t) = (t \, ds + s \, dt) \wedge (e^t \, dt) = te^t \, ds \wedge dt,$$

whence $d(F^*\eta) = F^*(d\eta)$, as claimed.

(b)(iii) We have

$$\omega \wedge \eta = -xy \, dx \wedge dy$$

and

$$\begin{aligned} F^*(\omega \wedge \eta) &= -(ste^t)\, d(st) \wedge d(e^t) = -(ste^t)(te^t)\, ds \wedge dt \\ &= -st^2e^{2t} \, ds \wedge dt. \end{aligned}$$

(c)(i) Using the facts that

$$dx \wedge dx = dy \wedge dy = dz \wedge dz = 0 \tag{1}$$

and

$$dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy, \tag{2}$$

we compute that

$$\begin{aligned}
d\omega &= d\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) \wedge dy \wedge dz + d\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) \wedge dz \wedge dx + \\
&\quad + d\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) \wedge dx \wedge dy \\
&= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dx \wedge dy \wedge dz + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) dy \wedge dz \wedge dx + \\
&\quad + \frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dz \wedge dx \wedge dy \\
&= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\, dx \wedge dy \wedge dz + \frac{-2y^2+x^2+z^2}{(x^2+y^2+z^2)^{5/2}}\, dx \wedge dy \wedge dz + \\
&\quad + \frac{-2z^2+x^2+y^2}{(x^2+y^2+z^2)^{5/2}}\, dx \wedge dy \wedge dz \\
&= 0.
\end{aligned}$$

Consider now the smooth function

$$f\colon N \to \mathbb{R}, \ (x,y,z) \mapsto \sqrt{x^2+y^2+z^2}$$

and observe that $\eta = df$. Therefore,

$$d\eta = 0.$$

(c)(ii) Using (1) and (2) we compute that

$$\begin{aligned}
\omega \wedge \eta &= \frac{x^2}{(x^2+y^2+z^2)^2}\, dy \wedge dz \wedge dx + \frac{y^2}{(x^2+y^2+z^2)^2}\, dz \wedge dx \wedge dy + \\
&\quad + \frac{z^2}{(x^2+y^2+z^2)^2}\, dx \wedge dy \wedge dz \\
&= \frac{1}{x^2+y^2+z^2}\, dx \wedge dy \wedge dz.
\end{aligned}$$

Moreover, since $d\eta = 0$ by (c)(i), we immediately infer that

$$\eta \wedge d\eta = 0.$$

(c)(iii) Since

$$du \wedge du = dv \wedge dv = 0, \quad du \wedge dv = -dv \wedge du$$

and

$$d\left(\sqrt{1-u^2-v^2}\right) = -\frac{u}{\sqrt{1-u^2-v^2}}\, du - \frac{v}{\sqrt{1-u^2-v^2}}\, dv,$$

we compute that

$$\begin{aligned} F^*\omega &= \frac{-u^2}{\sqrt{1-u^2-v^2}}\,dv\wedge du + \frac{-v^2}{\sqrt{1-u^2-v^2}}\,dv\wedge du + \sqrt{1-u^2-v^2}\,du\wedge dv \\ &= \frac{1}{\sqrt{1-u^2-v^2}}\,du\wedge dv. \end{aligned}$$

Moreover, since $d\eta = 0$ by (c)(i), we have

$$F^*(d\eta) = 0.$$

(c)(iv) Note that both $d(F^*\omega)$ and $F^*(d\omega)$ are 3-forms on the 2-dimensional manifold M. Hence they are both equal to 0; in particular, we have

$$d(F^*\omega) = F^*(d\omega) = 0.$$

Exercise 3: Let (r, θ) be polar coordinates on the right half-plane $H = \{(x, y) \mid x > 0\}$. Compute the polar coordinate expression for the smooth 1-form $x\,dy - y\,dx \in \Omega^1(\mathbb{R}^2)$ and for the smooth 2-form $dx \wedge dy \in \Omega^2(\mathbb{R}^2)$.

[Hint: Think of the change of coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ as the coordinate expression for the identity map of H, but using (r, θ) as coordinates for the domain and (x, y) as coordinates for the codomain.]

Solution: We have

$$\begin{aligned} \mathrm{Id}^*(x\,dy - y\,dx) &= r\cos\theta\, d(r\sin\theta) - r\sin\theta\, d(r\cos\theta) \\ &= r\cos\theta(\sin\theta\,dr + r\cos\theta\,d\theta) - r\sin\theta(\cos\theta\,dr - r\sin\theta\,d\theta) \\ &= r^2\cos^2\theta\,d\theta + r^2\sin^2\theta\,d\theta \\ &= r^2\,d\theta \end{aligned}$$

and

$$\begin{aligned} \mathrm{Id}^*(dx\wedge dy) &= d(r\cos\theta)\wedge d(r\sin\theta) \\ &= (\cos\theta\,dr - r\sin\theta\,d\theta)\wedge(\sin\theta\,dr + r\cos\theta\,d\theta) \\ &= r\cos^2\theta\,dr\wedge d\theta - r\sin^2\theta\,d\theta\wedge dr \\ &= r\,dr\wedge d\theta, \end{aligned}$$

since

$$dr\wedge dr = 0 = d\theta\wedge d\theta \quad\text{and}\quad dr\wedge d\theta = -d\theta\wedge dr.$$

Exercise 4: Consider the smooth 2-form

$$\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$$

on $\mathbb{R}^3$ with standard coordinates (x, y, z).

(a) Compute ω in spherical coordinates for $\mathbb{R}^3$ defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

(b) Compute $d\omega$ in spherical coordinates.

(c) Consider the inclusion map $\iota\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and compute the pullback $\iota^*\omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota^*\omega$ is nowhere zero.

Solution:

(a) We have

$$\begin{aligned} dx &= d(\rho \sin\varphi \cos\theta) = \sin\varphi \cos\theta \; d\rho + \rho \cos\varphi \cos\theta \; d\varphi - \rho \sin\varphi \sin\theta \; d\theta, \\ dy &= d(\rho \sin\varphi \sin\theta) = \sin\varphi \sin\theta \; d\rho + \rho \cos\varphi \sin\theta \; d\varphi + \rho \sin\varphi \cos\theta \; d\theta, \\ dz &= d(\rho \cos\varphi) = \cos\varphi \; d\rho - \rho \sin\varphi \; d\varphi, \end{aligned}$$

so we compute that

$$\begin{aligned} dy \wedge dz &= \rho^2 \sin^2\varphi \cos\theta \; d\varphi \wedge d\theta + \rho \sin\varphi \cos\varphi \cos\theta \; d\theta \wedge d\rho - \rho \sin\theta \; d\rho \wedge d\varphi, \\ dz \wedge dx &= \rho^2 \sin^2\varphi \sin\theta \; d\varphi \wedge d\theta + \rho \sin\varphi \cos\varphi \sin\theta \; d\theta \wedge d\rho + \rho \cos\theta \; d\rho \wedge d\varphi, \\ dx \wedge dy &= \rho^2 \cos\varphi \sin\varphi \; d\varphi \wedge d\theta - \rho \sin^2\varphi \; d\theta \wedge d\rho. \end{aligned}$$

By combining these expressions, we thus obtain

$$\begin{aligned} \omega &= x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy \\ &= (\rho^3 \sin^3\varphi \cos^2\theta + \rho^3 \sin^3\varphi \sin^2\theta + \rho^3 \cos^2\varphi \sin\varphi) \; d\varphi \wedge d\theta \\ &\quad + \underbrace{(\rho^2 \sin^2\varphi \cos\varphi \cos^2\theta + \rho^2 \sin^2\varphi \cos\varphi \sin^2\theta - \rho^2 \sin^2\varphi \cos\varphi)}_{=0} \; d\theta \wedge d\rho \\ &\quad + \underbrace{(-\rho^2 \sin\varphi \sin\theta \cos\theta + \rho^2 \sin\varphi \sin\theta \cos\theta)}_{=0} \; d\rho \wedge d\varphi \\ &= \rho^3 \sin\varphi \underbrace{(\sin^2\varphi \cos^2\theta + \sin^2\varphi \sin^2\theta + \cos^2\varphi)}_{=1} \; d\varphi \wedge d\theta \\ &= \rho^3 \sin\varphi \; d\varphi \wedge d\theta. \end{aligned}$$

(b) Since

$$d\big(\rho^3 \sin\varphi\big) = 3\rho^2 \sin\varphi \; d\rho + \rho^3 \cos\varphi \; d\varphi,$$

we obtain

$$d\omega = d\big(\rho^3 \sin\varphi\big) \wedge d\varphi \wedge d\theta = 3\rho^2 \sin\varphi \, d\rho \wedge d\varphi \wedge d\theta.$$

Another way to compute $d\omega$ would be to note that

$$d\omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy = 3 \, dx \wedge dy \wedge dz.$$

For the standard top differential form $dx \wedge dy \wedge dz$ on $\mathbb{R}^3$, a change of coordinates induces a factor given by the Jacobian determinant of the transformation. You may remember or look up (or compute) that the determinant of the Jacobian of the spherical coordinate transformation (see part (c) below) is $\rho^2 \sin\varphi$, so we obtain $d\omega = 3\rho^2 \sin\varphi \, d\rho \wedge d\varphi \wedge d\theta$.

(c) We just have to put $\rho = 1$ in the result of part (a). To justify precisely what is going on, let us spell this out in detail. The change from standard to spherical coordinates is provided by the diffeomorphism

$$\begin{aligned} G\colon \mathbb{R}_{>0} \times (0,\pi) \times (0,2\pi) &\to U \subseteq \mathbb{R}^3 \\ (\rho, \varphi, \theta) &\mapsto (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi), \end{aligned}$$

where

$$U = \mathbb{R}^3 \setminus \big\{(x, 0, z) \in \mathbb{R}^3 \mid x \geq 0, z \in \mathbb{R}\big\}.$$

So what we computed in part (a) above is $G^*(\omega|_U)$. Note that spherical coordinates on the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ are provided by the diffeomorphism

$$\begin{aligned} F\colon (0,\pi) \times (0,2\pi) &\to V \subseteq \mathbb{S}^2 \\ (\varphi, \theta) &\mapsto (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi), \end{aligned}$$

where

$$V = \mathbb{S}^2 \cap U.$$

If we denote by j the embedding

$$\begin{aligned} j\colon (0,\pi) \times (0,2\pi) &\to \mathbb{R}_{>0} \times (0,\pi) \times (0,2\pi) \\ (\varphi, \theta) &\mapsto (1, \varphi, \theta), \end{aligned}$$

then this is precisely set up so that $G \circ j = \iota \circ F$. What we actually want to compute is $F^*\iota^*(\omega|_U)$, and this is given by

$$\begin{aligned} F^*\iota^*(\omega|_U) &= (\iota \circ F)^*(\omega|_U) = (G \circ j)^*(\omega|_U) = j^*G^*(\omega|_U) \\ &= j^*(\rho^3 \sin\varphi \, d\varphi \wedge d\theta) \\ &= \sin\varphi \, d\varphi \wedge d\theta. \end{aligned}$$

(d) Since $\sin\varphi \neq 0$ for $\varphi \in (0,\pi)$, we infer that $F^*\iota^*(\omega|_U) = \sin\varphi \, d\varphi \wedge d\theta$ is nowhere vanishing on $(0,\pi) \times (0,2\pi)$. As F is an isomorphism, it follows that $\iota^*(\omega|_U) = (\iota^*\omega)|_V$ is nowhere vanishing on V: this is the set of all points on $\mathbb{S}^2$ except for the ones lying on the closed semi-circle on the xz-plane passing through $(1,0,0), (0,0,1), (0,0,-1)$. To conclude, note that we can do the exact same calculations for spherical coordinates around the x- and y-axes (just as we did before for spherical coordinates around the z-axis) in order to cover the whole $\mathbb{S}^2$, and we eventually deduce that $\iota^*\omega$ is nowhere zero on the whole $\mathbb{S}^2$, as desired.

Exercise 5:

(a) *Exterior derivative of a smooth* 1*-form*: Show that for any smooth 1-form ω and any smooth vector fields X and Y on a smooth manifold M it holds that

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X, Y]\big).$$

(b) Let M be a smooth n-manifold, let (E_i) be a smooth local frame for M and let (ε^i) be the dual coframe. For each i, denote by b^i_{jk} the component functions of the exterior derivative of ε^i in this frame, and for each j, k, denote by c^i_{jk} the component functions of the Lie bracket $[E_j, E_k]$:

$$d\varepsilon^i = \sum_{j<k} b^i_{jk}\, \varepsilon^j \wedge \varepsilon^k \quad \text{and} \quad [E_j, E_k] = c^i_{jk}\, E_i.$$

Show that $b^i_{jk} = -c^i_{jk}$.

Solution:

(a) Let $p \in M$ be arbitrary. Choose local coordinates $\big(U, (x^i)\big)$ around p and write

$$\omega = \sum_i c_i\, dx^i, \qquad X = \sum_i f_i \frac{\partial}{\partial x^i}, \qquad Y = \sum_i g_i \frac{\partial}{\partial x^i}.$$

Then

$$\begin{aligned}
[d\omega(X,Y)](p) = (d\omega)_p(X_p, Y_p) &= \sum_i [(dc_i)_p \wedge (dx^i)_p](X_p, Y_p) \\
&= \sum_i \big[(dc_i)_p(X_p)(dx^i)_p(Y_p) - (dc_i)_p(Y_p)(dx^i)_p(X_p)\big] \\
&= \sum_{i,j} \left[g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) - f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p) \right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\big[X\big(\omega(Y)\big)\big](p) &= \sum_i [X(c_i g_i)](p) = \sum_i g_i(p)[X(c_i)](p) + c_i(p)[X(g_i)](p) \\
&= \sum_{i,j} \left[g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p) \right]
\end{aligned}$$

and

$$\begin{aligned}
\big[Y\big(\omega(X)\big)\big](p) &= \sum_i [Y(c_i f_i)](p) = \sum_i f_i(p)[Y(c_i)](p) + c_i(p)[Y(f_i)](p) \\
&= \sum_{i,j} \left[f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) g_j(p) \frac{\partial f_i}{\partial x^j}(p) \right]
\end{aligned}$$

as well as

$$\big[\omega\big([X,Y]\big)\big](p) = \left[\sum_{i,j} c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p) - c_i(p) g_j(p) \frac{\partial f_i}{\partial x^j}(p) \right],$$

where we used *Proposition 7.13*. By combining these expressions, we obtain

$$\begin{aligned}
\big[X\big(\omega(Y)\big) - Y\big(\omega(X)\big)\big](p) &= \left(\sum_{i,j} g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p)\right) \\
&\quad - \left(\sum_{i,j} f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) g_j(p) \frac{\partial f_i}{\partial x^j}(p)\right) \\
&= \left(\sum_{i,j} g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) - f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p)\right) \\
&\quad + \left(\sum_{i,j} c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p) - c_i(p) g_j(p) \frac{\partial f_i}{\partial x^j}(p)\right) \\
&= [d\omega(X,Y)](p) + \big[\omega\big([X,Y]\big)\big](p).
\end{aligned}$$

(b) Let us compute $d\varepsilon^i(E_j, E_k)$ for some i, j, k with $j < k$. By part (a) we obtain

$$\begin{aligned}
d\varepsilon^i(E_j, E_k) &= E_j\big(\varepsilon^i(E_k)\big) - E_k\big(\varepsilon^i(E_j)\big) - \varepsilon^i\big([E_j, E_k]\big) \\
&= \underbrace{E_j(\delta_{ik})}_{=0} - \underbrace{E_k(\delta_{ij})}_{=0} - c^i_{jk} = -c^i_{jk},
\end{aligned}$$

where in the last step we used that a derivation evaluated at a constant function gives 0. On the other hand, we have

$$d\varepsilon^i(E_j, E_k) = \sum_{j'<k'} b^i_{j'k'} \left[\varepsilon^{j'} \wedge \varepsilon^{k'}\right](E_j, E_k) = b^i_{jk},$$

where we used that $\varepsilon^{j'} \wedge \varepsilon^{k'} = \varepsilon^{(j',k')}$; see [*Multilinear Algebra, Lemma C.20*(c) and *Proposition C.25*(c)]. Hence, $b^i_{jk} = -c^i_{jk}$.

Remark.

(1) *Exercise* 5(b) shows that the exterior derivative is in a certain sense dual to the Lie bracket. In particular, it shows that if we know all the Lie brackets of basis vector fields in a smooth local frame, we can compute the exterior derivatives of the dual covector fields, and vice versa.

(2) There is an analogue of *Exercise* 5(a) for smooth k-forms as well, which is referred to as the *invariant formula for the exterior derivative* in the literature. Specifically, if $\omega \in \Omega^k(M)$, then for any $X_1, \ldots, X_k \in \mathfrak{X}(M)$ it holds that

$$\begin{aligned}
d\omega(X_1, \ldots, X_{k+1}) &= \sum_{1 \le i \le k+1} (-1)^{i-1} X_i\big(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})\big) + \\
&\quad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}\big),
\end{aligned}$$

where the hats indicate omitted arguments. It is worthwhile to mention that the above formula can be used to give an *invariant* definition of d, as well as an alternative proof of *Theorem 8.27* on the existence, uniqueness, and properties of d.