EPFL

Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 13

Exercise 1:
Let $F\colon M \to N$ be a smooth map. Prove the following assertions:

(a) $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) It holds that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart $\big(V, (y^i)\big)$ for N, we have

$$F^* \left({\sum_I}' \omega_I \, dy^{i_1} \wedge \ldots \wedge dy^{i_k} \right) = {\sum_I}' (\omega_I \circ F) \, d\left(y^{i_1} \circ F\right) \wedge \ldots \wedge d\left(y^{i_k} \circ F\right).$$

Exercise 2 (to be submitted by Thursday, 18.12.2025, 16:00):

(a) Let V be a finite-dimensional real vector space and let $\omega^1, \ldots, \omega^k \in V^*$. Show that the covectors $\omega^1, \ldots, \omega^k$ are linearly dependent if and only if $\omega^1 \wedge \ldots \wedge \omega^k = 0$.

(b) Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2, \ (s,t) \mapsto (st, e^t)$$

and the smooth 1-forms

$$\omega = x \, dy - y \, dx, \ \eta = x \, dy \in \Omega^1(\mathbb{R}^2).$$

(Observe that we denote the standard coordinates on the source by (s,t) and the standard coordinates on the target by (x,y).)

(i) Compute $d\eta$ and $F^*\eta$.

(ii) Verify by direct computation that $d(F^*\eta) = F^*(d\eta)$.

(iii) Compute $\omega \wedge \eta$ and $F^*(\omega \wedge \eta)$.

(c) Consider the smooth manifolds

$$M = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\} \quad \text{and} \quad N = \mathbb{R}^3 \setminus \{0\},$$

the smooth map

$$F \colon M \to N, \ (u, v) \mapsto \left(u, v, \sqrt{1 - u^2 - v^2}\right),$$

and the differential forms

$$\omega = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \in \Omega^2(N)$$

and

$$\eta = \frac{x \, dx + y \, dy + z \, dz}{(x^2 + y^2 + z^2)^{1/2}} \in \Omega^1(N).$$

(i) Compute $d\omega$ and $d\eta$.

(ii) Compute $\omega \wedge \eta$ and $\eta \wedge d\eta$.

(iii) Compute $F^*\omega$ and $F^*(d\eta)$.

(iv) Verify that $d(F^*\omega) = F^*(d\omega)$.

Exercise 3:
Let (r, θ) be polar coordinates on the right half-plane $H = \{(x, y) \mid x > 0\}$. Compute the polar coordinate expression for the smooth 1-form $x \, dy - y \, dx \in \Omega^1(\mathbb{R}^2)$ and for the smooth 2-form $dx \wedge dy \in \Omega^2(\mathbb{R}^2)$.

[Hint: Think of the change of coordinates $(x, y) = (r \cos \theta, r \sin \theta)$ as the coordinate expression for the identity map of H, but using (r, θ) as coordinates for the domain and (x, y) as coordinates for the codomain.]

Exercise 4:
Consider the smooth 2-form

$$\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$$

on $\mathbb{R}^3$ with standard coordinates (x, y, z).

(a) Compute ω in spherical coordinates for $\mathbb{R}^3$ defined by

$$(x, y, z) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi).$$

(b) Compute $d\omega$ in spherical coordinates.

(c) Consider the inclusion map $\iota \colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and compute the pullback $\iota^*\omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota^*\omega$ is nowhere zero.

Exercise 5:

(a) *Exterior derivative of a smooth* 1-*form*: Show that for any smooth 1-form ω and any smooth vector fields X and Y on a smooth manifold M it holds that

$$d\omega(X,Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X,Y]\big).$$

(b) Let M be a smooth n-manifold, let (E_i) be a smooth local frame for M and let (ε^i) be the dual coframe. For each i, denote by b^i_{jk} the component functions of the exterior derivative of ε^i in this frame, and for each j, k, denote by c^i_{jk} the component functions of the Lie bracket $[E_j, E_k]$:

$$d\varepsilon^i = \sum_{j<k} b^i_{jk}\, \varepsilon^j \wedge \varepsilon^k \quad \text{and} \quad [E_j, E_k] = c^i_{jk}\, E_i.$$

Show that $b^i_{jk} = -c^i_{jk}$.