Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 12 – Solutions

Exercise 1 (*Smoothness criteria for covector fields*)**:** Let M be a smooth manifold and let $\omega\colon M \to T^*M$ be a rough covector field on M. Prove that the following assertions are equivalent:

(a) ω is smooth.

(b) In every smooth coordinate chart the component functions of ω are smooth.

(c) Every point of M is contained in some smooth coordinate chart in which ω has smooth component functions.

(d) For every smooth vector field X on M, the function $\omega(X)\colon M \to \mathbb{R}$ is smooth on M.

(e) For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X)\colon U \to \mathbb{R}$ is smooth on U.

[Hint: Try proving $(a) \implies (b) \implies (c) \implies (a)$ and $(c) \implies (d) \implies (e) \implies (b)$.]

Solution:

(a) $\implies$ (b): Suppose that ω is smooth. Let $\big(U,(x^i)\big)$ be a smooth chart for M and consider the corresponding smooth chart $\big(\pi^{-1}(U), \big((x^i),(\xi_i)\big)\big)$ for T^*M. It is characterized by sending $\xi_i\,\lambda^i|_p$ to $\big((x^i(p)),(\xi_i)\big)$, where $p \in U$ and $(\lambda_i|_p)$ is the dual basis of $(\partial/\partial x^i|_p)$. The component functions of ω with respect to the smooth chart $\big(U,(x^i)\big)$ are by definition the functions $\omega_i\colon U \to \mathbb{R}$ determined by

$$\omega_p = \sum_i \omega_i(p)\cdot\lambda^i|_p\,,\ p \in U.$$

Therefore, the coordinate representation $\hat{\omega}$ of ω with respect to these charts on U and $\pi^{-1}(U)$ is the map

$$\begin{aligned}\hat{\omega}\colon \hat{U} &\to \hat{U}\times\mathbb{R}^n\\ \hat{x} &\mapsto \big(\hat{x}, \big(\omega_i\circ\varphi^{-1}(\hat{x})\big)\big)\,.\end{aligned}$$

Since by hypothesis ω, and thus also $\hat{\omega}$, is smooth, we conclude that each $\omega_i \circ \varphi^{-1}$, and thus each ω_i itself, is smooth.

(b) $\implies$ (c): Immediate.

(c) $\implies$ (a): By hypothesis there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of M such that for each $\alpha \in A$, the covector field ω has smooth component functions in the chart $(U_\alpha, \varphi_\alpha)$. By the computation in $(a) \implies (b)$ we see that the coordinate representation of ω with respect to the smooth charts $\big(U_\alpha, \varphi_\alpha\big)$ for M and $\big(\pi^{-1}(U_\alpha), (\varphi_\alpha, (\xi_{\alpha,i})\big)$ for T^*M is smooth. Hence, ω is smooth.

(c) $\implies$ (d): Let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ be an atlas for which ω has smooth component functions $\omega_{\alpha,i}$, and write $\varphi_\alpha = (x^i_\alpha)$. Let $X^{\alpha,i}$ be the component functions of X on U_α, which are smooth by *Proposition 7.2*. Then, for any $p \in U_\alpha$, we have

$$\omega(X)(p) = \omega_p(X_p) = \sum_i \sum_j \omega_{\alpha,i}(p) X^{\alpha,j}(p) \underbrace{\lambda^i|_p \left(\left.\frac{\partial}{\partial x^j_\alpha}\right|_p \right)}_{=\delta^i_j}$$

$$= \sum_i \omega_{\alpha,i}(p) X^{\alpha,i}(p)\,,$$

as $(\lambda^i|_p)$ is the dual basis of $(\partial/\partial x^i_\alpha|_p)$. Since all functions $\omega_{\alpha,i}$ and $X_{\alpha,i}$ are smooth, we conclude that $\omega(X)|_{U_\alpha}$ is smooth. As $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is an atlas for M, it follows from *Proposition 2.9*(a) that $\omega(X)$ is smooth.

(d) $\implies$ (e): Let U be an open subset of M and let X be a smooth vector field on U. Let $p \in U$ and let (U_p, φ_p) be a smooth chart for M containing p. Let $\overline{V_p} \subseteq U_p$ be the preimage of a compact ball centered at $\varphi_p(p)$, and let V_p be its interior. Let $\psi_p \colon M \to \mathbb{R}$ be a smooth bump function with support in U_p such that $\psi_p|_{\overline{V_p}} \equiv 1$. Then the map $\psi_p X \colon M \to TM$ defined by

$$(\psi_p X)_q = \begin{cases} \psi_p(q) X_q & \text{if } q \in U, \\ 0 & \text{otherwise,} \end{cases}$$

is a smooth global vector field; indeed, it is smooth on U and on $M \setminus \operatorname{supp}(\psi_p)$ (as it is 0 on this set), which is an open cover of M by construction. Hence, $\omega(\psi_p X)$ is smooth by assumption. But then $\omega(X)|_{V_p} = \omega(\psi_p X)|_{V_p}$ is also smooth by *Proposition 2.9*(b). We conclude that there exists an open cover $\{V_p\}_{p \in U}$ of U such that $\omega(X)|_{V_p}$ is smooth for all $p \in U$, and thus $\omega(X) \colon U \to \mathbb{R}$ is smooth on U by *Proposition 2.9*(a).

(e) $\implies$ (b): Let $\big(U, (x^i)\big)$ be a smooth chart for M and let ω_i be the component functions of ω with respect to this chart. By applying (e) to the smooth vector field $\partial/\partial x^i \colon U \to \mathbb{R}$, we infer that the function $\omega(\partial/\partial x^i) \colon U \to \mathbb{R}$ is smooth. But this is equal to the function $\omega_i \colon U \to \mathbb{R}$, since for any $p \in U$ we have

$$\omega\left(\frac{\partial}{\partial x^i}\right)(p) = \sum_j \omega_j(p) \cdot \underbrace{\lambda^j|_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right)}_{=\delta^j_i} = \omega_i(p).$$

Therefore, the component functions ω_i of ω with respect to the given smooth chart $\big(U, (x^i)\big)$ are smooth.

Remark. The above arguments for $(d) \implies (e)$ and $(e) \implies (b)$ yield in particular the following: two (rough) covector fields $\omega, \omega' \colon M \to T^*M$ are equal if and only if $\omega(X) = \omega'(X)$ for all smooth global vector fields X on M.

Exercise 2: Let M be a smooth n-manifold. Show that TM is a smoothly trivial vector bundle if and only if T^*M is a smoothly trivial vector bundle.

Solution: Assume first that TM is a smoothly trivial vector bundle over M. Then there exists a smooth global trivialization $\Phi \colon TM \to M \times \mathbb{R}^n$, which corresponds to a smooth global frame $\{X_1, \dots, X_n\}$ by [*Exercise Sheet* 9*, Exercise* 5]. Then the global coframe $\{\omega_1, \dots, \omega_n\}$ dual to $\{X_1, \dots, X_n\}$ is smooth by *Lemma 8.7* and corresponds to a smooth global trivialization $\Psi \colon T^*M \to M \times \mathbb{R}^n$ by [*Exercise Sheet* 9*, Exercise* 5]. Therefore, T^*M is a smoothly trivial vector bundle over M.

Assume now that T^*M is a smoothly trivial vector bundle. Argue similarly, we conclude that TM is a smoothly trivial vector bundle.

Exercise 3 (*Properties of the differential*)**:** Let M be a smooth manifold and let $f, g \in C^\infty(M)$. Prove the following assertions:

(a) If $a, b \in \mathbb{R}$, then $d(af + bg) = a\, df + b\, dg$.

(b) $d(fg) = f\, dg + g\, df$.

(c) $d(f/g) = (g\, df - f\, dg)/g^2$ on the set where $g \neq 0$.

(d) If $J \subseteq \mathbb{R}$ is an interval containing the image of f and if $h \colon J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)\, df$.

(e) If f is constant, then $df = 0$. Conversely, if $df = 0$, then f is constant on each connected component of M.

Solution:

(a) Fix $a, b \in \mathbb{R}$ and $p \in M$. For any $v \in T_pM$ we have

$$\begin{aligned} d(af + bg)_p(v) &= v(af + bg) = a\, v(f) + b\, v(g) \\ &= a\, df_p(v) + b\, dg_p(v) \\ &= \big(a\, df_p + b\, dg_p\big)(v). \end{aligned}$$

Therefore,

$$d(af + bg)_p = a\, df_p + b\, dg_p,$$

which yields the statement, since $p \in M$ was arbitrary.

(b) Fix $p \in M$. For any $v \in T_pM$ we have

$$\begin{aligned} d(fg)_p(v) &= v(fg) = f(p)\, vg + g(p)\, vf \\ &= f(p)\, dg_p(v) + g(p)\, df_p(v) \\ &= \big(f(p)\, dg_p + g(p)\, df_p\big)(v). \end{aligned}$$

Therefore,

$$d(fg)_p = f(p)\, dg_p + g(p)\, df_p,$$

which yields the statement, since $p \in M$ was arbitrary.

Note: We may also argue somewhat differently as follows (the same also applies for (a) above, and this method will be used in (c) below as well): Let X be a smooth global vector field on M. For any $p \in M$ we have

$$d(fg)(X)(p) = X_p(fg) = f(p)\, X_p(g) + g(p)\, X_p(f) = (f\, dg)(X)(p) + (g\, df)(X)(p).$$

Therefore,

$$d(fg)(X) = (f\, dg)(X) + (g\, df)(X)$$

for any smooth global vector field X, which yields the statement.

(c) Let $U \coloneqq M \setminus g^{-1}(0)$. Let X be a smooth vector field on U. Given $p \in U$, note that

$$0 = X_p(1) = X_p\big(g \cdot (1/g)\big) = g(p) X_p(1/g) + \big(1/g(p)\big) X_p(g),$$

which yields

$$X_p(1/g) = -X_p(g)/\big(g(p)^2\big).$$

Therefore,

$$d(1/g)(X)(p) = X_p(1/g) = -X_p(g)/\big(g(p)^2\big) = \big(-(dg)/g^2\big)(X)(p)$$

for all X and p, which implies that

$$d(1/g) = -(dg)/g^2\,.$$

It follows that

$$d(f/g) \overset{(b)}{=} (1/g)\, df + f\, d(1/g) = (1/g)\, df - (f/g^2)\, dg = (g\, df - f\, dg)/g^2\,,$$

as desired.

(d) Fix $p \in M$ and $v \in T_pM$. Write $v = v^i \frac{\partial}{\partial x^i}\big|_p$ and note that

$$\left.\frac{\partial}{\partial x^i}\right|_p (h \circ f) = \frac{\partial (h \circ f)}{\partial x^i}(p) = h'\big(f(p)\big) \frac{\partial f}{\partial x^i}(p) = h'\big(f(p)\big) \left.\frac{\partial}{\partial x^i}\right|_p f$$

by the chain rule. Therefore,

$$\begin{aligned} d(h \circ f)_p(v) = v(h \circ f) &= \left(v^i \left.\frac{\partial}{\partial x^i}\right|_p \right) (h \circ f) \\ &= v^i\, h'\big(f(p)\big) \left.\frac{\partial}{\partial x^i}\right|_p f = h'\big(f(p)\big)\, vf \\ &= (h' \circ f)(p)\, df_p(v). \end{aligned}$$

Since $v \in T_pM$ was arbitrary, we infer that $d(h \circ f)_p = (h' \circ f)(p)\, df_p$, and since $p \in M$ was arbitrary, we conclude that $d(h \circ f) = (h' \circ f)\, df$, as desired.

Note: We may alternatively argue as follows: Let X be a smooth global vector field on M and let $p \in M$ be arbitrary. To avoid confusion, denote by $df_p\colon T_pM \to T_{f(p)}\mathbb{R}$ the differential of f at $p \in M$ as a linear map between tangent spaces, and by $d^{\text{cov}} f$ the covector field determined by f. They are related as follows: for every $p \in M$ and $v \in T_pM$, we have

$$d^{\text{cov}} f_p(v) = [df_p(v)](\text{Id}_{\mathbb{R}}).$$

This follows from the fact that the natural identifaction of $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$ is provided by evaluation at $\text{Id}_{\mathbb{R}}$. Therefore, if $p \in M$ and $v \in T_pM$ are arbitrary, then we have

$$\begin{aligned} d^{\text{cov}}(h \circ f)_p(v) &= [d(h \circ f)_p(v)](\text{Id}_{\mathbb{R}}) = \left[dh_{f(p)}\big(df_p(v)\big)\right](\text{Id}_{\mathbb{R}}) \\ &= h'\big(f(p)\big) \cdot [df_p(v)](\text{Id}_{\mathbb{R}}) = h'\big(f(p)\big) \cdot d^{\text{cov}} f_p(v), \end{aligned}$$

where we used that for any $t \in J$, the differential $dh_t\colon T_tJ \to T_{h(t)}\mathbb{R}$ is the map given by scalar multiplication with $h'(t)$. As $p \in M$ and $v \in T_pM$ were arbitrary, we conclude that $d^{\text{cov}}(h \circ f) = (h' \circ f)\, d^{\text{cov}} f$.

(e) In view of the fact that the differential of f as defined in *Chapter 3* (i.e., as a linear map $df_p\colon T_pM \to T_p\mathbb{R}$) and as defined in *Chapter 8* (i.e., as a linear map $df_p\colon T_pM \to \mathbb{R}$) is the same object (due to the canonical identification between $\mathbb{R}$ and $T_p\mathbb{R}$), the assertion is simply a special case of [*Exercise Sheet* 5*, Exercise* 1(*b*)].

Exercise 4:

(a) *Derivative of a function along a curve*: Let M be a smooth manifold, $\gamma\colon J \to M$ be a smooth curve, and $f\colon M \to \mathbb{R}$ be a smooth function. Show that the derivative of $f \circ \gamma\colon J \to \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\big(\gamma'(t)\big).$$

(b) Let M be a smooth manifold and let $f \in C^\infty(M)$. Show that $p \in M$ is a critical point of f if and only if $df_p = 0$.

(c) Consider the smooth manifold

$$M := \big\{(x, y) \in \mathbb{R}^2 \mid x > 0\big\}$$

and the smooth function

$$f\colon M \to \mathbb{R},\ (x, y) \mapsto \frac{x}{x^2 + y^2}.$$

Compute the coordinate representation for df and determine the set of all points $p \in M$ at which $df_p = 0$.

Solution:

(a) Using the definitions, for any $t \in J$ we have

$$df_{\gamma(t)}\big(\gamma'(t)\big) = \gamma'(t)\, f = d\gamma\left(\frac{d}{dt}\bigg|_t\right)(f) = \frac{d}{dt}\bigg|_t (f \circ \gamma) = (f \circ \gamma)'(t).$$

(b) Since the differential df_p is a linear map with codomain the 1-dimensional $\mathbb{R}$-vector space $T_p\mathbb{R} \cong \mathbb{R}$, it is surjective if and only if there exists $v \in T_pM \setminus \{0\}$ such that $df_p(v) \in \mathbb{R} \setminus \{0\} \cong T_p\mathbb{R} \setminus \{0\}$. Therefore, $p \in M$ is a critical point of f if and only if df_p is not surjective if and only if $df_p = 0$ (i.e., the zero linear map).

(c) Given a point $p = (x_0, y_0) \in M$, the differential df_p of f at p is represented in coordinates (x, y) by the row matrix D_p whose components are the partial derivatives of f at $p = (x_0, y_0)$; namely,

$$D_p = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right) = \left(\frac{y_0^2 - x_0^2}{(x_0^2 + y_0^2)^2}, \frac{-2x_0y_0}{(x_0^2 + y_0^2)^2} \right).$$

In view of part (b), to find the points $p \in M$ at which $df_p = 0$, we have to solve the system

$$(\Sigma) : \begin{cases} y^2 - x^2 = 0 \\ -2xy = 0 \end{cases}$$

under the restriction that $x > 0$. It is straightforward to see that (Σ) has no solutions $(x, y) \in M$; in other words,

$$\{p \in M \mid df_p = 0\} = \emptyset.$$

Exercise 5 (to be submitted):

(a) Consider the smooth map

$$F \colon \mathbb{R}^2 \to \mathbb{R}^2, \ (s, t) \mapsto (st, e^t)$$

and the smooth covector field

$$\omega = x\, dy - y\, dx \in \mathfrak{X}^*(\mathbb{R}^2).$$

Compute $F^*\omega$.

(b) Consider the function

$$f \colon \mathbb{R}^3 \to \mathbb{R}, \ (x, y, z) \mapsto x^2 + y^2 + z^2$$

and the map

$$F \colon \mathbb{R}^2 \to \mathbb{R}^3, \ (u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).[1]$$

Compute $F^*(df)$ and $d(f \circ F)$ separately, and verify that they are equal.

(c) Let M be a compact, connected, smooth manifold of dimension $n > 0$. Show that every exact smooth covector field on M vanishes at least at two points of M.

[1] Note that F is the inverse of the stereographic projection from the north pole $N \in \mathbb{S}^2$; see [*Exercise Sheet* 2, *Exercise* 6].

Solution:

(a) We have

$$\begin{aligned} F^*\omega &= (x\circ F)\, d\,(y\circ F) - (y\circ F)\, d\,(x\circ F) \\ &= (st)\, d\,(e^t) - \ (e^t)\, d\,(st) \\ &= ste^t\, dt - e^t(s\, dt + t\, ds) \\ &= (-te^t)\, ds + se^t(t-1)\, dt. \end{aligned}$$

(b) On the one hand, by *Exercise* 3 we obtain

$$df = d(x^2+y^2+z^2) = 2x\, dx + 2y\, dy + 2z\, dz,$$

and since

$$\begin{aligned} d\,(x\circ F) &= d\left(\frac{2u}{u^2+v^2+1}\right) = \frac{2(u^2+v^2+1)-4u^2}{(u^2+v^2+1)^2}\, du + \frac{-4uv}{(u^2+v^2+1)^2}\, dv, \\ d\,(y\circ F) &= d\left(\frac{2v}{u^2+v^2+1}\right) = \frac{-4uv}{(u^2+v^2+1)^2}\, du + \frac{2(u^2+v^2+1)-4v^2}{(u^2+v^2+1)^2}\, dv, \\ d\,(z\circ F) &= d\left(\frac{u^2+v^2-1}{u^2+v^2+1}\right) = \frac{4u}{(u^2+v^2+1)^2}\, du + \frac{4v}{(u^2+v^2+1)^2}\, dv, \end{aligned}$$

we compute that

$$\begin{aligned} F^*(df) &= 2(x\circ F)\, d\,(x\circ F) + 2(y\circ F)\, d\,(y\circ F) + 2(z\circ F)\, d\,(z\circ F) \\ &= 2\,\frac{2u}{u^2+v^2+1}\, d\left(\frac{2u}{u^2+v^2+1}\right) + 2\,\frac{2v}{u^2+v^2+1}\, d\left(\frac{2v}{u^2+v^2+1}\right) + \\ &\quad + 2\,\frac{u^2+v^2-1}{u^2+v^2+1}\, d\left(\frac{u^2+v^2-1}{u^2+v^2+1}\right) \\ &= \frac{\big(8u(u^2+v^2+1)-16u^3\big) - 16uv^2 + 8u(u^2+v^2-1)}{(u^2+v^2+1)^3}\, du + \\ &\quad + \frac{-16u^2v + \big(8v(u^2+v^2+1)-16v^3\big) + 8v(u^2+v^2-1)}{(u^2+v^2+1)^3}\, dv \\ &= 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (f\circ F)(u,v) &= \left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2 + \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2 \\ &= \frac{(u^2+v^2+1)^2}{(u^2+v^2+1)^2} \\ &= 1, \end{aligned}$$

whence $d(f\circ F) = 0$ according to *Exercise* 3(e).

(c) Let $\omega \in \mathfrak{X}^*(M)$ be exact and let $f \in C^\infty(M)$ such that $\omega = df$. Since M is compact, f attains its minimum at a point $p \in M$ and its maximum at a point $q \in M$, and since df is represented in coordinates by the gradient of (the coordinate representation of) f, we have $df_p = 0 = df_q$. Note also that if $p = q$, then f is constant, and thus $df = 0$ by *Exercise* 3(e).