EPFL

Differential Geometry II - Smooth Manifolds
Winter Term 2026/2027
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Exercise Sheet 10

Exercise 1: (a) Let G be a Lie group. Show that the vector space of left invariant vector fields can be identified with the tangent space at the identity.

(b) Compute the Lie bracket for GL_n explicitly using the identification of part (a) and $T_{\mathrm{id}}\mathrm{GL}_n \simeq \mathrm{Mat}_n(\mathbb{R})$.

Exercise 2:
Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon J \to M$ be an integral curve of V. Prove the following assertions:

(a) *Rescaling lemma*: For any $a \in \mathbb{R}$, the curve

$$\widetilde{\gamma}\colon \widetilde{J} \to M,\ t \mapsto \gamma(at)$$

is an integral curve of the vector field $\widetilde{V} \coloneqq aV$ on M, where $\widetilde{J} \coloneqq \{t \in \mathbb{R} \mid at \in J\}$.

(b) *Translation lemma*: For any $b \in \mathbb{R}$, the curve

$$\widehat{\gamma}\colon \widehat{J} \to M,\ t \mapsto \gamma(t+b)$$

is also an integral curve of V on M, where $\widehat{J} \coloneqq \{t \in \mathbb{R} \mid t+b \in J\}$.

Exercise 3: Compute the flow of each of the following smooth vector fields on $\mathbb{R}^2$:

(i) $U = y\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$.

(ii) $V = x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y}$.

(iii) $W = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

Exercise 4:
Let $\theta\colon \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. Show that the infinitesimal generator V of θ is a smooth vector field on M, and that each curve $\theta^{(p)}\colon \mathbb{R} \to M$ is an integral curve of V.

Definition. Let V be a (rough) vector field on a smooth manifold M. A point $p \in M$ is called a *singular point* of V if $V_p = 0 \in T_pM$; otherwise, it is called a *regular point* of V.

Exercise 5:
Let V be a smooth vector field on a smooth manifold M and let $\theta\colon \mathfrak{D} \to M$ be the flow generated by V. Prove the following assertions:

(a) If $p \in M$ is a singular point of V, then $\mathfrak{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$.

(b) If $p \in M$ is a regular point of V, then $\theta^{(p)}\colon \mathfrak{D}^{(p)} \to M$ is a smooth immersion.

[Hint: Argue by contraposition and use the fundamental theorem on flows.]