Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 10

Exercise 1:

(a) Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Show that X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N, we have $X(f \circ F) = (Yf) \circ F$.

(b) Consider the smooth map

$$F\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$

and the smooth vector fields

$$X = \frac{d}{dt} \in \mathfrak{X}(\mathbb{R}) \quad \text{and} \quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2).$$

Show that X and Y are F-related.

(c) Let $F\colon M \to N$ be a diffeomorphism and let $X \in \mathfrak{X}(M)$. Prove that there exists a unique smooth vector field Y on N that is F-related to X. The vector field Y is denoted by F_*X and is called the *pushforward of X by F*.

(d) Consider the open submanifolds

$$M \coloneqq \big\{(x,y) \in \mathbb{R}^2 \mid y > 0 \ \text{ and } \ x + y > 0\big\} \subseteq \mathbb{R}^2$$

and

$$N \coloneqq \big\{(u,v) \in \mathbb{R}^2 \mid u > 0 \ \text{ and } \ v > 0\big\} \subseteq \mathbb{R}^2$$

and the map

$$F\colon M \to N,\ (x,y) \mapsto \big(x+y, \frac{x}{y} + 1\big).$$

(i) Show that F is a diffeomorpism and compute its inverse F^{-1}.

(ii) Compute the pushforward F_*X of the following smooth vector field X on M:

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x}\bigg|_{(x,y)}.$$

Exercise 2 (to be submitted by Thursday, 27.11.2025, 16:00):

(a) *Coordinate formula for the Lie bracket*: Let M be a smooth n-manifold and let

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$$

be the coordinate expressions for $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)$, respectively, in terms of some smooth local coordinates (x^i) for M. Show that the Lie bracket $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(b) Compute the Lie brackets $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$ of the coordinate vector fields $\partial/\partial x^i$ in any smooth chart $\big(U, (x^i)\big)$ for a given smooth n-manifold M.

(c) Compute the Lie bracket of each of the following pairs of smooth vector fields on $\mathbb{R}^3$:

(i) $X_1 = y \dfrac{\partial}{\partial z} - 2xy^2 \dfrac{\partial}{\partial y}$ and $Y_1 = \dfrac{\partial}{\partial y}$.

(ii) $X_2 = -y \dfrac{\partial}{\partial x} + x \dfrac{\partial}{\partial y}$ and $Y_2 = -z \dfrac{\partial}{\partial y} + y \dfrac{\partial}{\partial z}$.

(iii) $X_3 = -y \dfrac{\partial}{\partial x} + x \dfrac{\partial}{\partial y}$ and $Y_3 = y \dfrac{\partial}{\partial x} + x \dfrac{\partial}{\partial y}$.

(d) Consider the open submanifold

$$M := \big\{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\big\} \subseteq \mathbb{R}^2,$$

the map

$$F \colon M \to M,\ (x, y) \mapsto \left(xy,\ \frac{y}{x}\right),$$

and the smooth vector fields

$$X := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{and} \quad Y := y \frac{\partial}{\partial x}$$

on M.

(i) Show that F is a diffeomorphism, compute its Jacobian matrix $DF(x, y)$ at an arbitrary point $(x, y) \in M$, and determine its inverse F^{-1}.

(ii) Compute the pushforwards $F_* X$ and $F_* Y$ of X and Y, respectively.

(iii) Compute the Lie brackets $[X, Y]$ and $[F_* X, F_* Y]$.

Exercise 3 (*Properties of the Lie bracket*)**:**
Let M be a smooth manifold. Show that the Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:

(a) *Bilinearity*: For all $a, b \in \mathbb{R}$ we have

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(b) *Antisymmetry*:
$$[X, Y] = -[Y, X].$$

(c) *Jacobi identity*:
$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

(d) For all $f, g \in C^\infty(M)$ we have

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

Exercise 4:
Let $F\colon M \to N$ be a smooth map.

(a) *Naturality of the Lie bracket*: Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be smooth vector fields such that X_i is F-related to Y_i for $i \in \{1, 2\}$. Show that $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

(b) *Pushforwards of Lie brackets*: Assume that F is a diffeomorphism and consider $X_1, X_2 \in \mathfrak{X}(M)$. Show that $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.

Exercise 5:

(a) *Restricting smooth vector fields to submanifolds*: Let M be a smooth manifold, let S be an immersed submanifold of M, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. Prove the following assertions:

  (i) If $Y \in \mathfrak{X}(M)$ and if there is $X \in \mathfrak{X}(S)$ that is ι-related to Y, then $Y \in \mathfrak{X}(M)$ is tangent to S.

  (ii) If $Y \in \mathfrak{X}(M)$ is tangent to S, then there is a unique smooth vector field on S, denoted by $Y|_S$, which is ι-related to Y.

  [Hint: Determine first the candidate vector field on S and then use *Theorem 5.9* and *Proposition 5.20* to show that it is smooth.]

(b) *Lie brackets of smooth vector fields tangent to submanifolds*: Let M be a smooth manifold and let S be an immersed submanifold of M. If Y_1 and Y_2 are smooth vector fields on M that are tangent to S, then show that their Lie bracket $[Y_1, Y_2]$ is also tangent to S.