Solution Set 9

Problem 1: Convergence of random variables

Let $(X_n,\, n \geq 1)$ be independent random variables such that $X_n \sim \text{Bern}(1 - \frac{1}{(n+1)^\alpha})$, where $\alpha > 0$. Let us also define $Y_n = \prod_{j=1}^n X_j$ for $n \geq 1$.

a) What minimal condition on the parameter $\alpha > 0$ ensures that $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$?

Hint: Use the approximation $1 - x \simeq \exp(-x)$ for x small.

b) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$?

c) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\to} 0$ almost surely ?

Hint: If $Y_n = 0$, what can you deduce on Y_m for $m \geq n$?

Solution a) Let us compute for $\varepsilon > 0$:

$$\mathbb{P}(\{|Y_n - 0| > \varepsilon\}) \leq \mathbb{P}(\{Y_n > 0\}) = \mathbb{P}(\{Y_n = 1\}) = \prod_{j=1}^n \mathbb{P}(\{X_j = 1\})$$
$$= \prod_{j=1}^n \left(1 - \frac{1}{(j+1)^\alpha}\right) \simeq \exp\left(-\sum_{j=1}^n \frac{1}{(j+1)^\alpha}\right)$$

where the hint was used in the last (approximate) equality. If $\alpha > 1$, then $\sum_{j=1}^n \frac{1}{(j+1)^\alpha}$ converges to a fixed value $< +\infty$ as $n \to \infty$, so $\mathbb{P}(\{Y_n > 0\})$ does not converge to 0 as $n \to \infty$.

On the contrary, if $0 < \alpha \leq 1$, then $\sum_{j=1}^n \frac{1}{(j+1)^\alpha} \underset{n\to\infty}{\to} +\infty$, in which case $\mathbb{P}(\{Y_n > 0\}) \underset{n\to\infty}{\to} 0$, so $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$ in this case.

b) The answer is yes. Indeed, we have $\mathbb{E}((Y_n - 0)^2) = \mathbb{E}(Y_n^2) = \mathbb{P}(\{Y_n = 1\})$, so $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$ if and only if $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$.

c) The answer is again yes. Indeed, if for a given realization ω, $Y_n(\omega) = 0$, then $Y_m(\omega) = 0$ for every $m \geq n$, and therefore $\lim_{n\to\infty} Y_n(\omega) = 0$. This implies that

$$\mathbb{P}(\{\lim_{n\to\infty} Y_n = 0\}) \geq \mathbb{P}(\{Y_n = 0\})$$

for any fixed value of $n \geq 1$. If $0 < \alpha \leq 1$, we have seen in question a) that $\mathbb{P}(\{Y_n = 0\}) \underset{n\to\infty}{\to} 1$. So the above inequality implies that $Y_n \underset{n\to\infty}{\to} 0$ almost surely in this case.

Remark. Please note finally that when $\alpha > 1$, convergence in probability does not hold, so automatically in this case, quadratic convergence and almost sure convergence do not hold either.

Problem 2: Second B-C lemma

a) Show that if $(A_n,\, n \geq 1)$ are *independent* events in $\mathcal{F}$ and $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$, then

$$\mathbb{P}\Big(\bigcup_{n\geq 1} A_n\Big) = 1$$

Hints: - Start by observing that the statement is equivalent to $\mathbb{P}\left(\bigcap_{n\geq 1} A_n^c\right) = 0$.
- Use the inequality $1 - x \leq e^{-x}$, valid for all $x \in \mathbb{R}$.

b) From the same set of assumptions, reach the following stronger conclusion with a little extra effort:

$$\mathbb{P}(\{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}) = \mathbb{P}\Big(\bigcap_{N\geq 1}\bigcup_{n\geq N} A_n\Big) = 1$$

which is actually the statement of the *second Borel-Cantelli lemma.*

c) Let $(X_n,\, n \geq 1)$ be a sequence of *independent* random variables such that for some $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty$. What can you conclude on the almost sure convergence of the sequence X_n towards the limiting value 0?

d) Let $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that $\mathbb{P}(\{X_n = n\}) = p_n = 1 - \mathbb{P}(\{X_n = 0\})$ for $n \geq 1$. What minimal condition on the sequence $(p_n, n \geq 1)$ ensures that

d1) $X_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? d2) $X_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? d3) $X_n \underset{n\to\infty}{\to} 0$ almost surely?

e) Let $(Y_n,\, n \geq 1)$ be a sequence of independent random variables such that $Y_n \sim \text{Cauchy}(\lambda_n)$ for $n \geq 1$. What minimal condition on the sequence $(\lambda_n,\, n \geq 1)$ ensures that

e1) $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? e2) $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? e3) $Y_n \underset{n\to\infty}{\to} 0$ almost surely?

Solution a) By independence, we obtain

$$\mathbb{P}\left(\bigcap_{n\geq 1} A_n^c\right) = \prod_{n\geq 1} \mathbb{P}(A_n^c) = \prod_{n\geq 1} (1 - \mathbb{P}(A_n)) \leq \prod_{n\geq 1} \exp(-\mathbb{P}(A_n)) = \exp\left(-\sum_{n\geq 1} \mathbb{P}(A_n)\right) = 0$$

where we have used the fact that $1 - x \leq \exp(-x)$ for $0 \leq x \leq 1$. Therefore, $\mathbb{P}\left(\bigcup_{n\geq 1} A_n\right) = 1$.

Note: The first equality above is "obviously true", but actually needs a proof (not required in the homework): if $(A_n,\, n \geq 1)$ is a countable sequence of independent events, then it holds that $\mathbb{P}(\cap_{n\geq 1} A_n) = \prod_{n\geq 1} \mathbb{P}(A_n)$. Here is why: define $B_n = \cap_{k=1}^n A_k$. Observe that $\cap_{n\geq 1} A_n = \cap_{n\geq 1} B_n$ and $B_n \supset B_{n+1}$ for every $n \geq 1$, so by the continuity property of $\mathbb{P}$,

$$\mathbb{P}(\cap_{n\geq 1} A_n) = \mathbb{P}(\cap_{n\geq 1} B_n) = \lim_{n\to\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} \prod_{k=1}^n \mathbb{P}(A_k) = \prod_{n\geq 1} \mathbb{P}(A_n)$$

b) By exactly the same argument as above, we can prove $\mathbb{P}\left(\bigcap_{n\geq N} A_n^c\right) = 0$, $\forall N \geq 1$, and we have seen in class that this holds true if and only if $\mathbb{P}\left(\bigcup_{N\geq 1}\bigcap_{n\geq N} A_n^c\right) = 0$, i.e. $\mathbb{P}\left(\bigcap_{N\geq 1}\bigcup_{n\geq N} A_n\right) = 1$.

c) If for some $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty$, then by part b), $\mathbb{P}(\{|X_n| \geq \varepsilon \text{ infinitely often}\}) = 1$. This says that almost sure convergence (towards the limiting value 0) of the sequence X_n does not hold, as for this convergence to hold, we would need exactly the opposite, namely that for every $\varepsilon > 0$, $\mathbb{P}(\{|X_n| \geq \varepsilon \text{ infinitely often}\}) = 0$.

d1) For any fixed $\varepsilon > 0$, $\mathbb{P}(\{|X_n| \geq \varepsilon\}) = p_n$ for sufficiently large n, so the minimal condition ensuring convergence in probability is simply $p_n \underset{n\to\infty}{\to} 0$ (said otherwise, $p_n = o(1)$).

d2) $\mathbb{E}((X_n - 0)^2) = n^2\, p_n$, so the minimal condition for L^2 convergence is $p_n = o(\frac{1}{n^2})$.

d3) Using the two Borel-Cantelli lemmas (both applicable here as the X_n are independent), we see that the minimal condition for almost sure convergence is $\sum_{n\geq 1} p_n < +\infty$, satisfied in particular if $p_n = O(n^{-1-\delta})$.

e1) We have in this case, for any fixed $\varepsilon > 0$:

$$\mathbb{P}(\{|Y_n| \geq \varepsilon\}) = 2\int_\varepsilon^{+\infty} dx\, \frac{1}{\pi}\, \frac{\lambda_n}{\lambda_n^2 + x^2} = \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan\left(\frac{\varepsilon}{\lambda_n}\right)\right) \underset{n\to\inf}{\to} 0$$

if and only if $\lambda_n \underset{n\to\infty}{\to} 0$.

e2) $\mathbb{E}(Y_n^2) = +\infty$ in all cases, so L^2 convergence does not hold.

e3) Observe first that by the change of variable $y = \lambda_n x$,

$$\mathbb{P}(\{|Y_n| \geq \varepsilon\}) = 2\int_\varepsilon^{+\infty} dy\, \frac{\lambda_n}{\pi\,(\lambda_n^2 + y^2)} = 2\int_{\varepsilon/\lambda_n}^{+\infty} dx\, \frac{1}{\pi(1+x^2)} \simeq 2\int_{\varepsilon/\lambda_n}^{+\infty} dx\, \frac{\lambda_n}{\pi x^2} = \frac{2\lambda_n}{\pi\,\varepsilon}$$

when λ_n is small. So the minimal condition for almost sure convergence is $\sum_{n\geq 1} \lambda_n < +\infty$, satisfied in particular if $\lambda_n = O(n^{-1-\delta})$.

Problem 3: Tail σ-field

a) Let $(X_n,\, n \geq 1)$ be a sequence of bounded i.i.d. random variables such that $\mathbb{E}(X_1) = 0$ and $\mathrm{Var}(X_1) = 1$, and let $S_n = X_1 + \ldots + X_n$ for $n \geq 1$. Show that the event

$$A = \left\{\frac{S_n}{n} \text{ converges}\right\}$$

belongs to the tail σ-field $\mathcal{T} = \cap_{n\geq 1}\sigma(X_n, X_{n+1}, \ldots)$ (implying that $\mathbb{P}(A) \in \{0,1\}$ by Kolomgorov's 0-1 law; but the law of large numbers tells you more in this case, namely that $\mathbb{P}(A) = 1$.).

b) Assume now that $(X_n,\, n \geq 1)$ is a sequence of bounded, *uncorrelated* and identically distributed random variables such that $\mathbb{E}(X_1) = 0$ and $\mathrm{Var}(X_1) = 1$. Under this more general assumption, Kolmogorov's 0-1 law may not necessarily hold. Prove it by exhibiting a sequence of random variables $(X_n,\, n \geq 1)$ satisfying these assumptions and an event $B \in \mathcal{T}$ such that $0 < \mathbb{P}(B) < 1$.

Solution a) Because the random variables X_n are identically distributed and bounded, it holds that there exists $M > 0$ such that $|X_n(\omega)| \leq M$ for all $n \geq 1$ and $\omega \in \Omega$. (*Note:* all this could hold with probability 1 instead of $\forall \omega \in \Omega$). So it holds that

$$\frac{S_n}{n} - \frac{1}{n}\sum_{j=2}^n X_j = \frac{X_1}{n} \underset{n\to\infty}{\to} 0 \quad \text{almost surely}$$

Likewise, it holds for any $k \geq 1$ that

$$\frac{S_n}{n} - \frac{1}{n}\sum_{j=k}^{n} X_j \underset{n\to\infty}{\to} 0 \quad \text{almost surely}$$

meaning that

$$A = \left\{\frac{S_n}{n} \text{ converges}\right\} = \left\{\frac{1}{n}\sum_{j=k}^{n} X_j \text{ converges}\right\} \in \sigma(X_k, X_{k+1}, \ldots)$$

As this holds for every $k \geq 1$, this proves that $A \in \mathcal{T}$.

b) We could actually show that $\mathbb{P}(A) = 1$ even when the X_n are just uncorrelated random variables, not necessarily independent. In order to find a counter-example, there remains therefore to find another event B.

To this end, let us consider the sequence $(Y_n,\, n \geq 1)$ of i.i.d. random variables such that $\mathbb{P}(\{Y_1 = +a\}) = 2/3$ and $\mathbb{P}(\{Y_1 = -2a\}) = 1/3$, and let Z be a random variable independent of the sequence $(Y_n,\, n \geq 1)$ such that $\mathbb{P}(\{Z = +1\}) = \mathbb{P}(\{Z = -1\}) = 1/2$. Let us finally define $X_n = Y_n \cdot Z$ for $n \geq 1$.

Choosing $a = 1/\sqrt{2}$, the random variables X_n have zero mean, unit variance and are uncorrelated, as

$$\mathbb{E}(X_n) = \mathbb{E}(Y_n) \cdot \mathbb{E}(Z) = 0, \quad \mathrm{Var}(X_n) = \mathbb{E}(X_n^2) = \mathbb{E}(Y_n^2) \cdot \mathbb{E}(Z^2) = \left(a^2\,\frac{2}{3} + 4a^2\,\frac{1}{3}\right) \cdot 1 = 2a^2 = 1$$

and for $n \neq m$:

$$\mathbb{E}(X_n \cdot X_m) = \mathbb{E}(Y_n \cdot Y_m) \cdot \mathbb{E}(Z^2) = \mathbb{E}(Y_n) \cdot \mathbb{E}(Y_m) \cdot 1 = (2a/3 - 2a/3)^2 = 0$$

Now, let $B = \{Z = +1\}$. The event B belongs to the tail σ-field $\mathcal{T}$ for the following reason: for any value of $n \geq 1$, the value of $X_n = Y_n \cdot Z$ determines the value of Z, as

$$Z = +1 \quad \text{if and only if} \quad X_n = +a \quad \text{or} \quad X_n = -2a$$

and likewise,

$$Z = -1 \quad \text{if and only if} \quad X_n = -a \quad \text{or} \quad X_n = +2a$$

So Z is measurable with respect to $\sigma(X_n) \subset \sigma(X_n, X_{n+1}, \ldots)$ for any $n \geq 1$, so Z is measurable with respect to $\mathcal{T} = \cap_{n\geq 1}\sigma(X_n, X_{n+1}, \ldots)$, i.e., $B = \{Z = +1\} \in \mathcal{T}$, but $\mathbb{P}(B) = 1/2 \notin \{0, 1\}$.

Problem 4: Another extension of the weak law of large numbers

Let $(T_n,\, n \geq 1)$ be another sequence of random variables, independent of the sequence $(X_n,\, n \geq 1)$, with all T_n taking values in the set of natural numbers $\mathbb{N}^* = \{1, 2, 3, \ldots\}$. Define

$$p_k^{(n)} = \mathbb{P}(\{T_n = k\}) \quad \text{for } n, k \geq 1 \qquad \left(\text{so } \sum_{k\geq 1} p_k^{(n)} = 1 \quad \forall n \geq 1\right)$$

a) Find a sufficient condition on the numbers $p_k^{(n)}$ guaranteeing that

$$\frac{X_1 + \ldots + X_{T_n}}{T_n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \mu \tag{1}$$

Hint: You should use the law of total probability here: if A is an event and the events $(B_k,\, k \geq 1)$ form a partition of Ω, then:

$$\mathbb{P}(A) = \sum_{k\geq 1} \mathbb{P}(A \,|\, B_k)\, \mathbb{P}(B_k)$$

b) Apply the above criterion to the following case: each T_n is the sum of two independent geometric random variables $G_{n1} + G_{n2}$, where both G_n are distributed as

$$\mathbb{P}(\{G_n = k\}) = q_n^{k-1}\,(1-q_n) \quad k \geq 1$$

where $0 < q_n < 1$.

b1) Compute first the distribution of T_n, as well as $\mathbb{E}(T_n)$, for each $n \geq 1$.

b2) What condition on the sequence $(q_n,\, n \geq 1)$ ensures that conclusion (1) holds?

Hint: Solving question b1) above may help you guessing what the answer to b2) should be.

Solution a) For $\varepsilon > 0$ and $n \geq 1$ fixed, let us compute, using the law of total probability:

$$\begin{aligned}
\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\}\right) &= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\} \,\Big|\, \{T_n = k\}\right) \cdot \mathbb{P}(\{T_n = k\}) \\
&= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\} \,\Big|\, \{T_n = k\}\right) \cdot \mathbb{P}(\{T_n = k\}) \\
&= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\}\right) \cdot p_k^{(n)}
\end{aligned}$$

by independence of T_n and the sequence $(X_n,\, n \geq 1)$. From the proof of the weak law of large numbers, we know that for every $k \geq 1$:

$$\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\}\right) \leq \frac{\sigma^2}{k\,\varepsilon^2}$$

so

$$\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\}\right) \leq \frac{\sigma^2}{\varepsilon^2} \sum_{k\geq 1} \frac{p_k^{(n)}}{k}$$

A sufficient condition ensuring convergence in probability is therefore: $\displaystyle\lim_{n\to\infty} \sum_{k\geq 1} \frac{p_k^{(n)}}{k} = 0$.

b1) Let us compute for $n \geq 1$ and $k \geq 2$: (noting that the probability is equal to zero for $k = 1$)

$$\begin{aligned}
p_k^{(n)} &= \mathbb{P}(\{T_n = k\}) = \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j,\, T_n = k\}) = \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j,\, G_{n2} = k-j\}) \\
&= \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j\}) \cdot \mathbb{P}(\{G_{n2} = k-j\}) = \sum_{j=1}^{k-1} q_n^{j-1}\,(1-q_n)\,q_n^{k-j-1}\,(1-q_n) \\
&= (k-1)\,q_n^{k-2}\,(1-q_n)^2
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbb{E}(T_n) &= \sum_{k\geq 2} k\,(k-1)\,q_n^{k-2}\,(1-q_n)^2 = \frac{\partial^2}{\partial z^2}\left(\sum_{k\geq 2} z^k\right)\Bigg|_{z=q_n} (1-q_n)^2 \\
&= \frac{\partial^2}{\partial z^2}\left(\frac{1}{1-z} - 1 - z\right)\Bigg|_{z=q_n} (1-q_n)^2 = \frac{2}{(1-q_n)^3}\,(1-q_n)^2 = \frac{2}{1-q_n}
\end{aligned}$$

Note: This result could also have been obtained using $\mathbb{E}(T_n) = \mathbb{E}(G_{n1}) + \mathbb{E}(G_{n2})$ together with the fact that a geometric random variable with parameter q has expectation $1/(1-q)$. [NB: geometric random variables with parameter q can be defined either on $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ (as it is the case here) or on $\mathbb{N} = \{0, 1, 2, \ldots\}$, as it was the case in Ex. 3 of Homework 4; their expectation is equal to $q/(1-q)$ in the latter case]

b2) From the above computations, we see that

$$\sum_{k \geq 1} \frac{p_k^{(n)}}{k} = \sum_{k \geq 2} \frac{k-1}{k} \, q_n^{k-2} \, (1-q_n)^2 \leq \sum_{k \geq 2} q_n^{k-2} \, (1-q_n)^2 = \frac{1}{1-q_n} \, (1-q_n)^2 = 1 - q_n$$

so convergence in probability occurs if $q_n \underset{n\to\infty}{\to} 1$. This is in accordance with the fact that $\mathbb{E}(T_n) \underset{n\to\infty}{\to} +\infty$ in this case (see part a).