Problem Set 9

Problem 1: Convergence of random variables

Let $(X_n,\, n \geq 1)$ be independent random variables such that $X_n \sim \text{Bern}(1 - \frac{1}{(n+1)^\alpha})$, where $\alpha > 0$.

Let us also define $Y_n = \prod_{j=1}^n X_j$ for $n \geq 1$.

a) What minimal condition on the parameter $\alpha > 0$ ensures that $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$?

Hint: Use the approximation $1 - x \simeq \exp(-x)$ for x small.

b) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$?

c) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\to} 0$ almost surely ?

Hint: If $Y_n = 0$, what can you deduce on Y_m for $m \geq n$?

Problem 2: Second B-C lemma

a) Show that if $(A_n,\, n \geq 1)$ are *independent* events in $\mathcal{F}$ and $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$, then

$$\mathbb{P}\Big(\bigcup_{n\geq 1} A_n\Big) = 1$$

Hints: - Start by observing that the statement is equivalent to $\mathbb{P}\left(\bigcap_{n\geq 1} A_n^c\right) = 0$.
- Use the inequality $1 - x \leq e^{-x}$, valid for all $x \in \mathbb{R}$.

b) From the same set of assumptions, reach the following stronger conclusion with a little extra effort:

$$\mathbb{P}(\{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}) = \mathbb{P}\Big(\bigcap_{N\geq 1} \bigcup_{n\geq N} A_n\Big) = 1$$

which is actually the statement of the *second Borel-Cantelli lemma.*

c) Let $(X_n,\, n \geq 1)$ be a sequence of *independent* random variables such that for some $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty$. What can you conclude on the almost sure convergence of the sequence X_n towards the limiting value 0?

d) Let $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that $\mathbb{P}(\{X_n = n\}) = p_n = 1 - \mathbb{P}(\{X_n = 0\})$ for $n \geq 1$. What minimal condition on the sequence $(p_n, n \geq 1)$ ensures that

d1) $X_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? d2) $X_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? d3) $X_n \underset{n\to\infty}{\to} 0$ almost surely?

e) Let $(Y_n,\, n \geq 1)$ be a sequence of independent random variables such that $Y_n \sim \text{Cauchy}(\lambda_n)$ for $n \geq 1$. What minimal condition on the sequence $(\lambda_n,\, n \geq 1)$ ensures that

e1) $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? e2) $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? e3) $Y_n \underset{n\to\infty}{\to} 0$ almost surely?

Problem 3: Tail σ-field

a) Let $(X_n,\, n \geq 1)$ be a sequence of bounded i.i.d. random variables such that $\mathbb{E}(X_1) = 0$ and $\text{Var}(X_1) = 1$, and let $S_n = X_1 + \ldots + X_n$ for $n \geq 1$. Show that the event

$$A = \left\{ \frac{S_n}{n} \text{ converges} \right\}$$

belongs to the tail σ-field $\mathcal{T} = \cap_{n\geq 1} \sigma(X_n, X_{n+1}, \ldots)$ (implying that $\mathbb{P}(A) \in \{0,1\}$ by Kolomgorov's 0-1 law; but the law of large numbers tells you more in this case, namely that $\mathbb{P}(A) = 1$.).

b) Assume now that $(X_n,\, n \geq 1)$ is a sequence of bounded, *uncorrelated* and identically distributed random variables such that $\mathbb{E}(X_1) = 0$ and $\text{Var}(X_1) = 1$. Under this more general assumption, Kolmogorov's 0-1 law may not necessarily hold. Prove it by exhibiting a sequence of random variables $(X_n,\, n \geq 1)$ satisfying these assumptions and an event $B \in \mathcal{T}$ such that $0 < \mathbb{P}(B) < 1$.

Problem 4: Another extension of the weak law of large numbers

Let $(T_n,\, n \geq 1)$ be another sequence of random variables, independent of the sequence $(X_n,\, n \geq 1)$, with all T_n taking values in the set of natural numbers $\mathbb{N}^* = \{1,2,3,\ldots\}$. Define

$$p_k^{(n)} = \mathbb{P}(\{T_n = k\}) \quad \text{for } n, k \geq 1 \qquad \left(\text{so } \sum_{k\geq 1} p_k^{(n)} = 1 \quad \forall n \geq 1 \right)$$

a) Find a sufficient condition on the numbers $p_k^{(n)}$ guaranteeing that

$$\frac{X_1 + \ldots + X_{T_n}}{T_n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \mu \tag{1}$$

Hint: You should use the law of total probability here: if A is an event and the events $(B_k,\, k \geq 1)$ form a partition of Ω, then:

$$\mathbb{P}(A) = \sum_{k\geq 1} \mathbb{P}(A \,|\, B_k)\, \mathbb{P}(B_k)$$

b) Apply the above criterion to the following case: each T_n is the sum of two independent geometric random variables $G_{n1} + G_{n2}$, where both G_n are distributed as

$$\mathbb{P}(\{G_n = k\}) = q_n^{k-1}\,(1 - q_n) \quad k \geq 1$$

where $0 < q_n < 1$.

b1) Compute first the distribution of T_n, as well as $\mathbb{E}(T_n)$, for each $n \geq 1$.

b2) What condition on the sequence $(q_n,\, n \geq 1)$ ensures that conclusion (1) holds?

Hint: Solving question b1) above may help you guessing what the answer to b2) should be.