Solution Set 8

Problem 1: Gaussian random variables

a) Let X_1, X_2 be two independent Gaussian random variables such that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$. Show, using characteristic functions or a result from the course, that $X_1 + X_2$ and $X_1 - X_2$ are also independent Gaussian random variables.

b) Let X_1, X_2 be two independent square-integrable random variables such that $X_1 + X_2, X_1 - X_2$ are also independent random variables. Show that X_1, X_2 are jointly Gaussian random variables such that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$.

Note. Part b), also known as Darmois-Skitovic's theorem, is considerably more challenging than part a)! Here are the steps to follow in order to prove the result (but please skip the first two).

Step 1.* (needs the dominated convergence theorem, which is outside of the scope of this course) If X is a square-integrable random variable, then ϕ_X is twice continuously differentiable.

Step 2.* (quite technical) Under the assumptions made, ϕ_{X_1} and ϕ_{X_2} have no zeros (so $\log \phi_{X_1}$ and $\log \phi_{X_2}$ are also twice continuously differentiable, according to the previous step).

Step 3. Let $f_1 = \log \phi_{X_1}$ and $f_2 = \log \phi_{X_2}$. Show that there exist functions g_1, g_2 satisfying

$$f_1(t_1 + t_2) + f_2(t_1 - t_2) = g_1(t_1) + g_2(t_2) \quad \forall t_1, t_2 \in \mathbb{R}$$

Step 4. If f_1, f_2 are twice continuously differentiable and there exist functions g_1, g_2 satisfying

$$f_1(t_1 + t_2) + f_2(t_1 - t_2) = g_1(t_1) + g_2(t_2) \quad \forall t_1, t_2 \in \mathbb{R}$$

then f_1, f_2 are polynomials of degree less than or equal to 2. *Hint:* differentiate!

Step 5. If X is square-integrable and $\log \phi_X$ is a polynomial of degree less than or equal to 2, then X is a Gaussian random variable.

Hint. If X is square-integrable, then you can take for granted that $\phi_X(0) = 1$, $\phi_X'(0) = i\mathbb{E}(X)$ and $\phi_X''(0) = -\mathbb{E}(X^2)$.

Step 6. From the course, deduce that X_1, X_2 are jointly Gaussian and that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$.

Solution a) Option 1: by the assumptions made, $\mathrm{Cov}(X_1 + X_2, X_1 - X_2) = \mathrm{Var}(X_1) + \mathrm{Cov}(X_2, X_1) - \mathrm{Cov}(X_1, X_2) - \mathrm{Var}(X_2) = \mathrm{Var}(X_1) - \mathrm{Var}(X_2) = 0$. Besides, as X_1, X_2 are independent Gaussian random variables, $X = (X_1, X_2)$ is a Gaussian random vector, so $(X_1 + X_2, X_1 - X_2)$ is also a Gaussian random vector whose components are uncorrelated, and therefore independent, by Proposition 6.8 of the course.

Option 2 is to show directly that

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)})\, \mathbb{E}(e^{it_2(X_1-X_2)}) \quad \forall t_1, t_2 \in \mathbb{R}$$

as this would imply independence of $X_1 + X_2$ and $X_1 - X_2$. We check indeed that

$$\begin{aligned}\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) &= \mathbb{E}(e^{i(t_1+t_2)X_1+i(t_1-t_2)X_2)}) \\ &= \mathbb{E}(e^{i(t_1+t_2)X_1})\, \mathbb{E}(e^{i(t_1-t_2)X_2}) = e^{i\mu_1(t_1+t_2)-\sigma_1^2(t_1+t_2)^2/2}\, e^{i\mu_2(t_1-t_2)-\sigma_2^2(t_1-t_2)^2/2}\end{aligned}$$

Because of the assumption made ($\sigma_1^2 = \sigma_2^2 = \sigma^2$), the above expression is further equal to

$$
\begin{aligned}
&= e^{i(\mu_1+\mu_2)t_1 + i(\mu_1-\mu_2)t_2 - \sigma^2(t_1^2+t_2^2)} = e^{i(\mu_1+\mu_2)t_1 - \sigma^2 t_1^2}\, e^{i(\mu_1-\mu_2)t_2 - \sigma^2 t_2^2} \\
&= \mathbb{E}(e^{it_1(X_1+X_2)})\, \mathbb{E}(e^{it_2(X_1-X_2)})
\end{aligned}
$$

which proves the claim.

b) 1. Skipped. Just note that closing our eyes, we could compute

$$\phi_X'(t) = i\,\mathbb{E}(X\,e^{itX}) \quad \text{and} \quad \phi_X''(t) = -\mathbb{E}(X^2 e^{itX}), \quad t \in \mathbb{R}$$

and deduce from there that indeed, if $\mathbb{E}(X^2) < +\infty$, then ϕ_X is twice continuously differentiable. As a by-product, we obtain the relation

$$\phi_X''(0) = -\mathbb{E}(X^2)$$

from the second formula evaluated in $t = 0$.

2. Skipped.

3. By the assumptions made, we obtain

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)})\, \mathbb{E}(e^{it_2(X_1-X_2)})$$

and also

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{i(t_1+t_2)X_1 + i(t_1-t_2)X_2}) = \phi_{X_1}(t_1+t_2)\, \phi_{X_2}(t_1-t_2)$$

so

$$\log \phi_{X_1}(t_1+t_2) + \log \phi_{X_2}(t_1-t_2) = \log \mathbb{E}(e^{it_1(X_1+X_2)}) + \log \mathbb{E}(e^{it_2(X_1-X_2)}) = g_1(t_1) + g_2(t_2)$$

proving the claim.

4. Differentiating first the equality with respect to t_1, we obtain

$$f_1'(t_1+t_2) + f_2'(t_1-t_2) = g_1'(t_1)$$

and then with respect to t_2:

$$f_1''(t_1+t_2) - f_2''(t_1-t_2) = 0$$

Setting $t_1 = t_2 = \frac{t}{2}$ leads to $f_1''(t) = f_2''(0)$, and setting $t_1 = -t_2 = \frac{t}{2}$ leads to $f_2''(t) = f_1''(0)$. As these equalities are satisfied for arbitrary $t \in \mathbb{R}$, this says that the second derivatives of both f_1 and f_2 are constant functions, therefore that both f_1 and f_2 are polynomials of degree less than or equal to 2.

5. The assumption is that $\log \phi_X(t) = at^2 + bt + c$ for $t \in \mathbb{R}$. Using the hint and writing $\mu = \mathbb{E}(X)$, $\sigma^2 = \mathrm{Var}(X)$, we obtain successively:

$$
\begin{aligned}
e^c = \phi_X(0) &= 1 && \text{so } c = 0 \\
b = \phi_X'(0) &= i\mu && \text{so } b = i\mu \\
2a + b^2 = \phi_X''(0) &= -\mathbb{E}(X^2) = -(\mu^2+\sigma^2) && \text{so } a = -\sigma^2/2
\end{aligned}
$$

Therefore, $\phi_X(t) = e^{i\mu t - \sigma^2 t^2/2}$, which is the characteristic function of a Gaussian.

6. As X_1, X_2 are independent and Gaussian, this implies that (X_1, X_2) is a Gaussian vector, i.e., that X_1, X_2 are jointly Gaussian. By the assumptions made, we also have

$$0 = \mathrm{Cov}(X_1+X_2, X_1-X_2) = \mathrm{Var}(X_1) + \mathrm{Cov}(X_2, X_1) - \mathrm{Cov}(X_1, X_2) - \mathrm{Var}(X_2) = \mathrm{Var}(X_1) - \mathrm{Var}(X_2)$$

so $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$ [note in passing that we did not use here the assumption that X_1 and X_2 are uncorrelated]. This finally completes the proof of the result stated in part b).