Problem Set 6

Problem 1: Inequalities

a) Let X be a square-integrable random variable such that $\mathbb{E}(X) = 0$ and $\mathrm{Var}(X) = \sigma^2$. Show that

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2} \quad \text{for } t > 0$$

Hint: You may try various versions of Chebyshev's inequality here, but not all of them work. A possibility is to use the function $\psi(x) = (x+b)^2$, where b is a free parameter to optimize (but watch out that only some values of $b \in \mathbb{R}$ lead to a function ψ that satisfies the required hypotheses).

b) Deduce from a) that for any square-integrable random variable X with expectation μ and variance σ^2, the following inequality holds:

$$\mathbb{P}(\{X \geq \mu + \sigma\}) \leq \frac{1}{2}$$

c) *Numerical application:* Check the inequality in b) for $X \sim \mathrm{Bern}(\frac{1}{2})$.

d) Let X be a square-integrable random variable such that $\mathbb{E}(X) > 0$. Show that

$$\mathbb{P}(\{X > t\}) \geq \frac{(\mathbb{E}(X) - t)^2}{\mathbb{E}(X^2)} \quad \forall 0 \leq t \leq \mathbb{E}(X)$$

Hint: Use first Cauchy-Schwarz' inequality with the random variables X and $Y = 1_{\{X>t\}}$.

e) Deduce from d) that for any square-integrable random variable X with expectation $\mu > 0$ and variance σ^2 satisfying $0 \leq \sigma \leq \mu$, the following inequality holds:

$$\mathbb{P}(\{X > \mu - \sigma\}) \geq \frac{\sigma^2}{\sigma^2 + \mu^2}$$

f) *Numerical application:* Check the inequality in e) for $X \sim \mathrm{Bern}(\frac{1}{2})$.

Solution

a) Using $\psi(x) = x^2$ or $\psi(x) = \sigma^2 + x^2$ in Chebyshev's inequality leads to respectively

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{t^2} \quad \text{and} \quad \mathbb{P}(\{X \geq t\}) \leq \frac{2\sigma^2}{\sigma^2 + t^2}$$

which is not what we want. Using the hint (with $b \geq 0$ in order to satisfy the hypotheses), we obtain

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\mathbb{E}((X+b)^2)}{(t+b)^2} = \frac{\sigma^2 + b^2}{(t+b)^2}$$

Optimizing over the parameter b, we find that best possible bound is obtained by setting $b^* = \frac{\sigma^2}{t}$ (which is non-negative), leading to

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

b) Applying the inequality in a) to the random variable $X - \mu$ (with the same variance σ^2), we obtain

$$\mathbb{P}(\{X - \mu \geq \sigma\}) \leq \frac{\sigma^2}{\sigma^2 + \sigma^2} = \frac{1}{2}$$

c) For $X \sim \text{Bern}(\frac{1}{2})$, we have $\mu = \sigma = \frac{1}{2}$, so indeed $\mathbb{P}(\{X \geq 1\}) = \frac{1}{2}$: the above inequality is an equality in this case.

d) Using Cauchy-Schwarz's inequality with the random variables X and $Y = 1_{\{X>t\}}$, we obtain

$$\mathbb{E}(X\, 1_{\{X>t\}})^2 \leq \mathbb{E}(X^2)\, \mathbb{P}(\{X > t\})$$

On the other hand, we have $\mathbb{E}(X\, 1_{\{X>t\}}) = \mathbb{E}(X) - \mathbb{E}(X\, 1_{\{X\leq t\}}) \geq \mathbb{E}(X) - t$, therefore the result.

e) The inequality follows directly from d).

f) For $X \sim \text{Bern}(\frac{1}{2})$, we have $\mu = \sigma = \frac{1}{2}$, so indeed $\mathbb{P}(\{X > 0\}) = \frac{1}{2}$: the above inequality is (also) an equality in this case.