Problem Set 7

Problem 1: Convolution

Let X_1, X_2 be two independent and identically distributed (i.i.d.) $\mathcal{N}(0,1)$ random variables. Compute the pdf of $X_1 + X_2$ (using convolution).

Solution By the formula seen in class, we have:

$$\begin{aligned} p_{X_1+X_2}(t) &= \int_{\mathbb{R}} dx_1 \, p_{X_1}(x_1) \, p_{X_2}(t-x_1) = \int_{\mathbb{R}} dx_1 \, \frac{1}{\sqrt{2\pi}} \exp(-x_1^2/2) \, \frac{1}{\sqrt{2\pi}} \exp(-(t-x_1)^2/2) \\ &= \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \int_{\mathbb{R}} dx_1 \, \frac{1}{\sqrt{2\pi}} \exp(tx_1 - x_1^2) \\ &= \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \int_{\mathbb{R}} dx_1 \, \frac{1}{\sqrt{2\pi}} \exp(-(x_1 - t/2)^2) \exp(t^2/4) \\ &= \frac{1}{\sqrt{4\pi}} \exp(-t^2/4) \int_{\mathbb{R}} dx_1 \, \frac{1}{\sqrt{\pi}} \exp(-(x_1 - t/2)^2) \end{aligned}$$

The integral on the right-hand side is equal to 1, as the integrand is the pdf of a $\mathcal{N}(t/2, 1/2)$ random variable, so we remain with

$$p_{X_1+X_2}(t) = \frac{1}{\sqrt{4\pi}} \exp(-t^2/4), \quad t \in \mathbb{R}$$

which shows that $X_1 + X_2$ is a $\mathcal{N}(0,2)$ random variable.

Problem 2: Moment generating function

The moment-generating function of a random variable X is defined for any $t \in \mathbb{R}$ as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

(Notice that it is similar but not equal to the characteristic function of X!) Let $X \sim \mathrm{Bi(n,p)}$ where, recall that, the Binomial distribution with parameters (n,p) measures the probability of k successes in n independent Bernoulli trials each with parameter p.

a) Prove that for every $a \in \mathbb{R}$ and $t > 0$,

$$\mathbb{P}(X \geq a) \leq e^{-ta} M_X(t).$$

b) Show that

$$M_X(t) = (pe^t + (1-p))^n.$$

c) Using the inequality in part a) and optimizing over all $t > 0$, show that for any fixed q such that $p < q < 1$,

$$\mathbb{P}(X \geq qn) \leq \left(\frac{p}{q}\right)^{qn} \left(\frac{1-p}{1-q}\right)^{(1-q)n}.$$

d) Using Markov inequality, show that

$$\mathbb{P}(X \geq qn) \leq \frac{p}{q}$$

and compare this inequality with the one in part c).

Solution

a) The result follows directly from the Chebyshev–Markov inequality with $\psi(x) = e^{tx}$.

b) We can write $X = \sum_{i=1}^{n} B_i$, where the B_i's are n i.i.d. Bernoulli(p) random variables. Then, for each B_i we have

$$\mathbb{E}(e^{tB_i}) = pe^t + 1 - p$$

so that we have

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \mathbb{E}\left(e^{t\sum_i B_i}\right) \\ &= \mathbb{E}\left(\prod_i e^{tB_i}\right) \\ &= \prod_i \mathbb{E}(e^{tB_i}) \\ &= (pe^t + 1 - p)^n. \end{aligned}$$

c) By applying the inequality in part (a) to X with $a = qn$, we get

$$\mathbb{P}(X \geq qn) \leq \left(\frac{pe^t + 1 - p}{e^{tq}}\right)^n.$$

Since y^n is an increasing function for $y > 0$, in order to optimize the right-hand side over t, we can substitute $z = e^t$ and optimize the function

$$\frac{pz + 1 - p}{z^q}, \qquad z > 0.$$

By taking the derivative and setting it equal to 0, we get

$$\frac{pz^q - qz^{q-1}(pz + 1 - p)}{z^{2q}} = 0 \iff pz - pqz - q(1-p) = 0 \iff z = \frac{q}{p} \cdot \frac{1-p}{1-q}.$$

Substituting $z = e^t$ in the right-hand side of the inequality leads to the result.

d) We have that

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_i B_i\right) = \sum_i \mathbb{E}(B_i) = np,$$

so that Markov inequality for $a = qn$ becomes

$$\mathbb{P}(X \geq qn) \leq \frac{\mathbb{E}(X)}{nq} = \frac{np}{nq} = \frac{p}{q}.$$

Note that the second inequality does not depend on n. This is in general bad. In fact, when n is large we expect X to concentrate around np (its expectation). Since $q > p$, we therefore expect that $\mathbb{P}(X \geq qn) \to 0$ when $n \to \infty$. This is indeed what we get from the first inequality: the right-hand side goes to 0 when $n \to \infty$. However, the second inequality is just a constant for every n, and therefore it is very loose when n is large.