Problem Set 7

Problem 1: Convolution

Let X_1, X_2 be two independent and identically distributed (i.i.d.) $\mathcal{N}(0,1)$ random variables. Compute the pdf of $X_1 + X_2$ (using convolution).

Problem 2: Moment generating function

The moment-generating function of a random variable X is defined for any $t \in \mathbb{R}$ as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

(Notice that it is similar but not equal to the characteristic function of X!) Let $X \sim \mathrm{Bi(n,p)}$ where, recall that, the Binomial distribution with parameters (n,p) measures the probability of k successes in n independent Bernoulli trials each with parameter p.

a) Prove that for every $a \in \mathbb{R}$ and $t > 0$,

$$\mathbb{P}(X \geq a) \leq e^{-ta} M_X(t).$$

b) Show that

$$M_X(t) = (pe^t + (1-p))^n.$$

c) Using the inequality in part a) and optimizing over all $t > 0$, show that for any fixed q such that $p < q < 1$,

$$\mathbb{P}(X \geq qn) \leq \left(\frac{p}{q}\right)^{qn} \left(\frac{1-p}{1-q}\right)^{(1-q)n}.$$

d) Using Markov inequality, show that

$$\mathbb{P}(X \geq qn) \leq \frac{p}{q}$$

and compare this inequality with the one in part c).