Problem Set 6

Problem 1: Inequalities

a) Let X be a square-integrable random variable such that $\mathbb{E}(X) = 0$ and $\mathrm{Var}(X) = \sigma^2$. Show that

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2} \quad \text{for } t > 0$$

Hint: You may try various versions of Chebyshev's inequality here, but not all of them work. A possibility is to use the function $\psi(x) = (x + b)^2$, where b is a free parameter to optimize (but watch out that only some values of $b \in \mathbb{R}$ lead to a function ψ that satisfies the required hypotheses).

b) Deduce from a) that for any square-integrable random variable X with expectation μ and variance σ^2, the following inequality holds:

$$\mathbb{P}(\{X \geq \mu + \sigma\}) \leq \frac{1}{2}$$

c) *Numerical application:* Check the inequality in b) for $X \sim \mathrm{Bern}(\frac{1}{2})$.

d) Let X be a square-integrable random variable such that $\mathbb{E}(X) > 0$. Show that

$$\mathbb{P}(\{X > t\}) \geq \frac{(\mathbb{E}(X) - t)^2}{\mathbb{E}(X^2)} \quad \forall 0 \leq t \leq \mathbb{E}(X)$$

Hint: Use first Cauchy-Schwarz' inequality with the random variables X and $Y = 1_{\{X>t\}}$.

e) Deduce from d) that for any square-integrable random variable X with expectation $\mu > 0$ and variance σ^2 satisfying $0 \leq \sigma \leq \mu$, the following inequality holds:

$$\mathbb{P}(\{X > \mu - \sigma\}) \geq \frac{\sigma^2}{\sigma^2 + \mu^2}$$

f) *Numerical application:* Check the inequality in e) for $X \sim \mathrm{Bern}(\frac{1}{2})$.