Differential Geometry II - Smooth Manifolds
Winter Term 2026/2027
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# Exercise Sheet 8

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**Exercise 1:** (a) Consider the map

$$F\colon \mathbb{R}^4 \to \mathbb{R}^2,\ (x,y,s,t) \mapsto (x^2+y,\ x^2+y^2+s^2+t^2+y).$$

Show that $(0,1) \in \mathbb{R}^2$ is a regular value of $F$, and that the level set $F^{-1}(0,1)$ is diffeomorphic to $\mathbb{S}^2$.

(b) Consider the smooth function

$$\Phi\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x^2 - y^2.$$

Given $c \in \mathbb{R}$, examine whether the corresponding level set $\Phi^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.

(c) Determine the regular values of the smooth function

$$\Psi\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x^2 - y^3$$

and draw the level sets corresponding to its critical values.

**Exercise 2:**

(a) Let $M$ be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Show that if $\Phi\colon U \to N$ is a local defining map for $S$, then it holds that

$$T_pS \cong \ker\left(d\Phi_p\colon T_pM \to T_{\Phi(p)}N\right) \quad \text{for every } p \in S \cap U.$$

(b) Let $M$ be a smooth manifold. Suppose that $S \subseteq M$ is a level set of a smooth submersion $\Phi = (\Phi_1, \ldots, \Phi_k)\colon M \to \mathbb{R}^k$. Show that a vector $v \in T_pM$ is tangent to $S$ if and only if $v\Phi_1 = \ldots = v\Phi_k = 0$.

**Exercise 3:** Recall that $\mathrm{GL}_n, \mathrm{SL}_n$, and $\mathrm{SO}_n$ are the group of real $n \times n$ matrices that are invertible, with determinant 1, and preserving the Euclidean inner product, respectively. Show that each of them are Lie groups and compute the tangent space at the identity. Check that for the tangent spaces, you obtain

- $\mathrm{Mat}_n$ the vector space of all $n \times n$ matrices,
- $\{X \in \mathrm{Mat}_n : \mathrm{tr}(X) = 0\}$ the vector space of traceless matrices, and
- $\{X \in \mathrm{Mat}_n : X^T = -X\}$ the vector space of antisymmetric matrices, respectively.

**Exercise 4:**

(a) Let $M$ be a smooth manifold, let $S \subseteq M$ be an immersed or embedded submanifold, and let $p \in S$. Show that a vector $v \in T_pM$ is in $T_pS$ if and only if there exists a smooth curve $\gamma\colon J \to M$ whose image is contained in $S$, and which is also smooth as a map into $S$, such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

(b) Let $M$ be a smooth manifold, let $S \subseteq M$ be an embedded submanifold and let $\gamma\colon J \to M$ be a smooth curve whose image happens to lie in $S$. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$.

(c) Give a counterexample to part (b) if $S \subseteq M$ is immersed but not embedded.

[Hint: Example 4.5(2).]