EPFL

Differential Geometry II - Smooth Manifolds
Winter Term 2026/2027
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Exercise Sheet 7

Exercise 1:

(a) *Sufficient conditions for properness*: Let X and Y be topological spaces and let $F\colon X \to Y$ be a continuous map. Prove the following assertions:

(i) If X is compact and Y is Hausdorff, then F is proper.

(ii) If F is a topological embedding with closed image, then F is proper.

(iii) If Y is Hausdorff and F has a continuous *left inverse*, i.e., a continuous map $G\colon Y \to X$ such that $G \circ F = \mathrm{Id}_X$, then F is proper.

(b) Let M be a smooth manifold and let S be an embedded submanifold of M. Show that S is properly embedded if and only if S is a closed subset of M.

(c) *Global graphs are properly embedded*: Let $f\colon M \to N$ be a smooth map between smooth manifolds. Show that the graph $\Gamma(f)$ of f is a properly embedded submanifold of $M \times N$.

Exercise 2:
Consider the smooth curve

$$\beta\colon (-\pi, \pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t, \sin t).$$

Show that its image is not an embedded submanifold of $\mathbb{R}^2$.

Exercise 3: (a) *Restricting the domain of a smooth map*: If $F\colon M \to N$ is a smooth map and if $S \subseteq M$ is an embedded submanifold, then the restriction $F|_S\colon S \to N$ is smooth.

(b) *Restricting the codomain of a smooth map*: Let M be a smooth manifold, let $S \subseteq M$ be an embedded submanifold, and let $G\colon N \to M$ be a smooth map whose image is contained in S. If G is a continuous map from N to S, then $G\colon N \to S$ is smooth.

(c) Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Then every smooth map $G\colon N \to M$ whose image is contained in S is also smooth as a map from N to S.

Exercise 4: Let M be a smooth n-manifold. Show that if $S \subseteq M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition, i.e., each point of S is contained in the domain of a smooth chart (U, φ) for M such that

$$S \cap U = \left\{(x^1, \ldots, x^n) \in U : x^{k+1} = c^{k+1}, \ldots, x^n = c^n\right\}$$

for some constants $c^{k+1}, \ldots, c^n$.