



Differential Geometry II - Smooth Manifolds

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Exercise Sheet 6

Exercise 1: Prove the following assertions:

- (a) A local diffeomorphism is open.
- (b) Any bijective local diffeomorphism is a diffeomorphism.

Exercise 2:

- (a) Prove the following assertions:
 - (i) A composition of smooth submersions is a smooth submersion.
 - (ii) A composition of smooth immersions is a smooth immersion.
 - (iii) A composition of smooth embeddings is a smooth embedding.
- (b) Show by means of a counterexample that a composition of smooth maps of constant rank need not have constant rank.
- (c) Let M and N be smooth manifolds and let $F: M \rightarrow N$ be a map. Prove the following assertions:
 - (i) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.
 - (ii) If $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Exercise 3:

Prove the following assertions:

- (a) The quotient map $\pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$ is smooth.
- (b) A map $F: \mathbb{RP}^n \rightarrow M$ to a smooth manifold M is smooth if and only if the composite map $F \circ \pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow M$ is smooth.

- (c) For any point $p \in \mathbb{R}^{n+1} \setminus \{0\}$, the differential $d\pi_p: T_p(\mathbb{R}^{n+1} \setminus \{0\}) \rightarrow T_{[p]}\mathbb{RP}^n$ is surjective (i.e., $\pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$ is a smooth submersion) and its kernel is the subspace generated by p .

Exercise 4: (a) Let $M_1, \dots, M_k$ be smooth manifolds, where $k \geq 2$. Show that each of the projection maps

$$\pi_i: M_1 \times \dots \times M_k \rightarrow M_i$$

is a smooth submersion.

- (b) Let $M_1, \dots, M_k$ be smooth manifolds, where $k \geq 2$. Choosing arbitrarily points $p_1 \in M_1, \dots, p_k \in M_k$, for each $1 \leq j \leq k$ consider the map

$$\iota_j: M_j \rightarrow M_1 \times \dots \times M_k, \quad x \mapsto (p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_k).$$

Show that each ι_j is a smooth embedding.

- (c) Show that the inclusion map $\iota: \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding, where $n \geq 1$.
 (d) Show that the map

$$G: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto ((2 + \cos 2\pi u) \cos 2\pi v, (2 + \cos 2\pi u) \sin 2\pi v, \sin 2\pi u)$$

is a smooth immersion.

Exercise 5 (Characteristic property of surjective smooth submersions):

Let $\pi: M \rightarrow N$ be a surjective smooth submersion. Prove the following assertion: For any smooth manifold P , a map $F: N \rightarrow P$ is smooth if and only if the composite map $F \circ \pi: M \rightarrow P$ is smooth.

$$\begin{array}{ccc} M & & \\ \pi \downarrow & \searrow^{F \circ \pi} & \\ N & \xrightarrow{F} & P \end{array}$$

Exercise 6 (The converse of the preceding exercise is false): Consider the map

$$\pi: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto xy.$$

Show that π is surjective and smooth, and that for each smooth manifold P , a map $F: \mathbb{R} \rightarrow P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.