



Differential Geometry II - Smooth Manifolds

Winter Term 2025/2026

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Exercise Sheet 6

Exercise 1 (to be submitted by Thursday, 30.10.2025, 16:00):

- (a) Let M be a topological manifold. Using facts from [Lee, Chapter 1, Section: Topological Properties of Manifolds](#), justify briefly the following assertion: There exists a countable, locally finite family $\mathfrak{U} = (U_i)_{i \in \mathbb{N}}$ of relatively compact open subsets of M such that $M = \bigcup_{i \in \mathbb{N}} U_i$.
- (b) Let N and $M_1, \dots, M_k$ be smooth manifolds, where $k \geq 2$, and let $F_i: N \rightarrow M_i$ be smooth maps, where $1 \leq i \leq k$. Show that the map

$$G: N \rightarrow \prod_{i=1}^k M_i, \quad x \mapsto (F_1(x), \dots, F_k(x))$$

is smooth and that its differential at any point $p \in N$ is of the form

$$(dG_p)(v) = (d(F_1)_p(v), \dots, d(F_k)_p(v)), \quad v \in T_p N.$$

- (c) Let M be a smooth manifold. Show that there exists a smooth map $f: M \rightarrow [0, +\infty)$ that is proper.

[Hint: Consider a function of the form $f = \sum_{i=1}^{+\infty} c_i \psi_i$, where $(\psi_i)_{i=1}^{+\infty}$ is a smooth partition of unity subordinate to a cover $\mathfrak{U}$ of M as in part (a) and $(c_i)_{i=1}^{+\infty}$ is an appropriate sequence of non-negative real numbers.]

- (d) Let $F: M \rightarrow N$ be an injective smooth immersion between smooth manifolds. Show that there exists a smooth embedding $G: M \rightarrow N \times \mathbb{R}$.

[Hint: Use parts (b) and (c).]

Exercise 2 (Characteristic property of surjective smooth submersions):

Let $\pi: M \rightarrow N$ be a surjective smooth submersion. Prove the following assertion: For any smooth manifold P , a map $F: N \rightarrow P$ is smooth if and only if the composite map $F \circ \pi: M \rightarrow P$ is smooth.

$$\begin{array}{ccc} M & & \\ \pi \downarrow & \searrow F \circ \pi & \\ N & \xrightarrow{F} & P \end{array}$$

Exercise 3:

Let M and N be smooth manifolds, and let $\pi: M \rightarrow N$ be a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of *Exercise 2*; in other words, assuming that $\tilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P , a map $F: \tilde{N} \rightarrow P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\tilde{N}$.

Exercise 4 (The converse of *Exercise 2* is false):

Consider the map

$$\pi: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto xy.$$

Show that π is surjective and smooth, and that for each smooth manifold P , a map $F: \mathbb{R} \rightarrow P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Exercise 5 (*Pushing smoothly to the quotient*):

Let $\pi: M \rightarrow N$ be a surjective smooth submersion. Prove the following assertion: If P is a smooth manifold and $F: M \rightarrow P$ is a smooth map that is constant on the fibers of π , then there exists a unique smooth map $\tilde{F}: N \rightarrow P$ such that $\tilde{F} \circ \pi = F$.

$$\begin{array}{ccc} M & & \\ \pi \downarrow & \searrow F & \\ N & \xrightarrow{\tilde{F}} & P \end{array}$$

Exercise 6 (*Uniqueness of smooth quotients*):

Let $\pi_1: M \rightarrow N_1$ and $\pi_2: M \rightarrow N_2$ be surjective smooth submersions that are constant on each other's fibers. Show that there exists a unique diffeomorphism $F: N_1 \rightarrow N_2$ such that $F \circ \pi_1 = \pi_2$:

$$\begin{array}{ccc} & M & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ N_1 & \xrightarrow{F} & N_2 \end{array}$$