

Differential Geometry II - Smooth Manifolds Winter Term 2025/2026 Lecturer: Dr. N. Tsakanikas

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Exercise Sheet 6

Exercise 1 (to be submitted by Thursday, 30.10.2025, 16:00):

- (a) Let M be a topological manifold. Using facts from Lee, Chapter 1, Section: Topological Properties of Manifolds, justify briefly the following assertion: There exists a countable, locally finite family $\mathfrak{U} = (U_i)_{i \in \mathbb{N}}$ of relatively compact open subsets of M such that $M = \bigcup_{i \in \mathbb{N}} U_i$.
- (b) Let N and M_1, \ldots, M_k be smooth manifolds, where $k \geq 2$, and let $F_i : N \to M_i$ be smooth maps, where $1 \leq i \leq k$. Show that the map

$$G \colon N \to \prod_{i=1}^k M_i, \ x \mapsto (F_1(x), \dots, F_k(x))$$

is smooth and that its differential at any point $p \in N$ is of the form

$$(dG_p)(v) = (d(F_1)_p(v), \dots, d(F_k)_p(v)), v \in T_pN.$$

(c) Let M be a smooth manifold. Show that there exists a smooth map $f: M \to [0, +\infty)$ that is proper.

[Hint: Consider a function of the form $f = \sum_{i=1}^{+\infty} c_i \psi_i$, where $(\psi_i)_{i=1}^{+\infty}$ is a smooth partition of unity subordinate to a cover \mathfrak{U} of M as in part (a) and $(c_i)_{i=1}^{+\infty}$ is an appropriate sequence of non-negative real numbers.]

(d) Let $F: M \to N$ be an injective smooth immersion between smooth manifolds. Show that there exists a smooth embedding $G: M \to N \times \mathbb{R}$.

[Hint: Use parts (b) and (c).]

Exercise 2 (Characteristic property of surjective smooth submersions):

Let $\pi: M \to N$ be a surjective smooth submersion. Prove the following assertion: For any smooth manifold P, a map $F: N \to P$ is smooth if and only if the composite map $F \circ \pi: M \to P$ is smooth.



Exercise 3:

Let M and N be smooth manifolds, and let $\pi\colon M\to N$ be a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of $Exercise\ 2$; in other words, assuming that \widetilde{N} represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P, a map $F\colon \widetilde{N}\to P$ is smooth if and only if $F\circ\pi$ is smooth, show that Id_N is a diffeomorphism between N and \widetilde{N} .

Exercise 4 (The converse of *Exercise 2* is false):

Consider the map

$$\pi \colon \mathbb{R}^2 \to \mathbb{R}, \ (x,y) \mapsto xy.$$

Show that π is surjective and smooth, and that for each smooth manifold P, a map $F \colon \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Exercise 5 (Pushing smoothly to the quotient):

Let $\pi \colon M \to N$ be a surjective smooth submersion. Prove the following assertion: If P is a smooth manifold and $F \colon M \to P$ is a smooth map that is constant on the fibers of π , then there exists a unique smooth map $\widetilde{F} \colon N \to P$ such that $\widetilde{F} \circ \pi = F$.

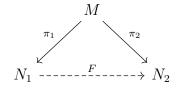
$$M$$

$$\pi \downarrow \qquad F$$

$$N \xrightarrow{\widetilde{F}} P$$

Exercise 6 (Uniqueness of smooth quotients):

Let $\pi_1: M \to N_1$ and $\pi_2: M \to N_2$ be surjective smooth submersions that are constant on each other's fibers. Show that there exists a unique diffeomorphism $F: N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$:



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