Midterm exam: solutions

Exercise 1. Quiz. (24 points) Answer each yes/no or open question below (1 pt) and provide a short justification (proof or counter-example) for your answer (2 pts).

a) Let the function $g\colon \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = 1_{\mathbb{Q}}(x)$. That is, $g(x)$ outputs one if x is a rational number, and zero otherwise. Is $g(x)$ Borel-measurable?

Solution: Yes, $g(x)$ Borel-measurable since $\mathbb{Q}$ is a Borel-measurable set. Using the definition in lecture we can check that $\{x \in \mathbb{R}\colon g(x) \in B\} \in \{\emptyset, \mathbb{Q}, \mathbb{Q}^c, \mathbb{R}\}$ for every $B \in \mathcal{B}(\mathbb{R})$. For example, $\{x \in \mathbb{R}\colon g(x) \in B\} = \mathbb{Q}$ if $1 \in B$ and $0 \notin B$.

b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = \{1, \ldots, 6\}$ and $\mathcal{F} = \sigma(\{\{1,2,3\}, \{1,3,5\}\})$. Given that $\mathbb{P}(\{1,2,3\}) = \mathbb{P}(\{1,3,5\}) = \mathbb{P}(\{2,5\}) = \frac{1}{2}$, what is $\mathbb{P}(\{4,6\})$?

Solution: We know that $\mathcal{F}$ is generated by the following atoms $\{\{1,3\}, \{2\}, \{5\}, \{4,6\}\}$. Using the additivity property of probability measures we can set up and solve a system of equations to find that every atom occurs with probability $\frac{1}{4}$, and specifically $\mathbb{P}(\{4,6\}) = \frac{1}{4}$.

c) Let $(\Omega, \mathcal{F})$ be a measurable space. Suppose $B \in \mathcal{F}$. Show that $\mathcal{G} = \{A \cap B\colon A \in \mathcal{F}\}$ is a σ-field of subsets of B. In other words, $(B, \mathcal{G})$ is also a measurable space.

Solution: To show that $\mathcal{G}$ is a σ-field on B we need to check the three axioms for the σ-field.

(i) $\emptyset, B \in \mathcal{G}$ since $B \cap B = B$ and $B \cap \emptyset = \emptyset$.

(ii) $G \in \mathcal{G}$ means that $G = A \cap B$ for some $A \in \mathcal{F}$. But then $G^c \in \mathcal{G}$ since $G^c = A^c \cap B$. Observe that here G^c is taken with respect to B.

(iii) If $G_n, n \geq 1$ is a sequence of subsets in B, then $\cup_{n=1}^{\infty} G_n \in \mathcal{G}$. Indeed, for some sequence $A_n, n \geq 1$ in $\mathcal{F}$ we have

$$\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (A_n \cap B) = \left(\bigcup_{n=1}^{\infty} A_n\right) \bigcap B \in \mathcal{G}.$$

d) Let X and Y be two random variables defined on the same probability space. Given $\mathbb{P}(X > Y) > 1/2$, is it possible that $\mathbb{E}(X) < \mathbb{E}(Y)$?

Solution: Yes. For an example, let X and Y be defined on $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbb{P}$ the Lebesgue measure. Then, let $X = 0.1$ and $Y = 1_{\{\omega \geq 0.6\}}$. In this case, $\mathbb{P}(X > Y) = 0.6$ while $\mathbb{E}(X) = 0.1 < 0.4 = \mathbb{E}(Y)$.

e) Let X and Y be two Bernoulli random variables defined on the same probability space. If X

and Y are uncorrelated, are they also independent?

Solution: Yes. We know that $\mathbb{E}(X) = 0 \times \mathbb{P}(X = 0) + 1 \times \mathbb{P}(X = 1) = \mathbb{P}(X = 1)$ and similarly, $\mathbb{E}(Y) = \mathbb{P}(Y = 1)$. On the other hand, $\mathbb{E}(XY) = 0 \times \mathbb{P}(X = 0 \cup Y = 0) + 1 \times \mathbb{P}(X = 1, Y = 1) = \mathbb{P}(X = 1, Y = 1)$. Since X and Y are uncorrelated, $\mathbb{P}(X = 1, Y = 1) = \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{P}(X = 1)\mathbb{P}(Y = 1)$. We also know that if two events $\{X = 1\}, \{Y = 1\}$ are independent, their complements $\{X = 0\}, \{Y = 0\}$ are also independent, i.e., $\mathbb{P}(X = 0, Y = 0) = \mathbb{P}(X = 0)\mathbb{P}(Y = 0)$. Thus, X and Y are independent.

f) A continuous random variable X has CDF $\mathbb{F}_X$. What is the CDF of $Y = aX + b$, where a and b are real constants?

Solution: There are three cases to consider:

If $a > 0$ then

$$\mathbb{F}_Y(t) = \mathbb{P}(\{Y \leq t\}) = \mathbb{P}(\{aX + b \leq t\}) = \mathbb{P}\left(\left\{X \leq \frac{1}{a}(t-b)\right\}\right) = \mathbb{F}_X\left(\frac{1}{a}(t-b)\right).$$

If $a < 0$ then

$$\mathbb{F}_Y(t) = \mathbb{P}(\{Y \leq t\}) = \mathbb{P}(\{aX + b \leq t\}) = \mathbb{P}\left(\left\{X \geq \frac{1}{a}(t-b)\right\}\right) = 1 - \mathbb{P}\left(\left\{X < \frac{1}{a}(t-b)\right\}\right)$$
$$= 1 - \mathbb{F}_X\left(\frac{1}{a}(t-b)\right).$$

If $a = 0$ then

$$\mathbb{F}_Y(t) = \mathbb{P}(\{Y \leq t\}) = \mathbb{P}(\{b \leq t\}) = \begin{cases} 0, & t < b \\ 1, & t \geq b \end{cases}$$

g) We flip a fair coin n times. Let A_{ij} be the event that ith and jth flips are the same. Show that the events $\{A_{ij} \colon 1 \leq i < j \leq n\}$ are pairwise independent but not independent.

Solution: First, consider the pair of events A_{ij} and A_{lm} where i, j, l, m are all distinct. In this case we have

$$\mathbb{P}(A_{ij} \cap A_{lm}) = \frac{1}{4} = \mathbb{P}(A_{ij})\,\mathbb{P}(A_{lm})$$

Secondly, consider the pair of events A_{ij} and A_{ik} with $i \neq k$. In this case

$$\mathbb{P}(A_{ij} \cap A_{ik}) = \frac{1}{4} = \mathbb{P}(A_{ij})\,\mathbb{P}(A_{ik})$$

Indeed, probability of jth and kth flips are independent. So, the probability of each being equal to ith flip is just $\frac{1}{2}$. Thus, this collection of events is pairwise indenedent.

The collection of events is not independent since

$$\mathbb{P}(A_{ij} \cap A_{ik} \cap A_{jk}) = \mathbb{P}(A_{ij} \cap A_{ik}) \neq \mathbb{P}(A_{ij})\,\mathbb{P}(A_{ik})\,\mathbb{P}(A_{jk}).$$

h) Let U be a random variable uniformly distributed on the interval $[0, 1]$. Let F be any CDF, and define $\phi(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}$ for $0 < u < 1$. What is the CDF of the random variable $\phi(U)$?

Solution: Since F is non-decreasing, if $F(z) \leq u$, then $\phi(u) \geq z$. On the other hand, since F is right-continuous, we have that $\inf\{x; F(x) \geq u\} = \min\{x; F(x) \geq u\}$; that is, the infimum is actually obtained. It follows that if $F(z) \geq u$, then $\phi(u) \leq z$. Hence, $\phi(u) \leq z$ if and only if $u \leq F(z)$. Since $0 \leq F(z) \leq 1$, we obtain that $\mathbf{P}(\phi(U) \leq z) = \mathbf{P}(U \leq F(z)) = F(z)$.

Exercise 2. (14 points) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbb{P}$ the Lebesgue measure on $\mathcal{B}([0,1])$.

a) Let the collection $\{I_1, \dots, I_4\}$ be a partition of Ω such that $I_i = \left[\frac{i-1}{4}, \frac{i}{4}\right)$ for $i \in \{1,2,3\}$ and $I_4 = \left[\frac{3}{4}, 1\right]$. Similarly, let $\{J_1, \dots, J_6\}$ be a partition of Ω such that $J_i = \left[\frac{i-1}{6}, \frac{i}{6}\right)$ for $i \in \{1, \dots, 5\}$ and $J_6 = \left[\frac{5}{6}, 1\right]$. Describe the σ-fields $\mathcal{F}_1 = \sigma(\{I_1, \dots, I_4\})$ and $\mathcal{F}_2 = \sigma(\{J_1, \dots, J_6\})$. How many elements does $\mathcal{F}_1$ and $\mathcal{F}_2$ have?

Solution: Since $\{I_1, \dots, I_4\}$ is a partition, these are also the atoms of $\mathcal{F}_1$. Thus, there are 4 atoms and $2^4 = 16$ elements in $\mathcal{F}_1$. Similarly, there are 6 atoms and $2^6 = 64$ elements in $\mathcal{F}_2$.

b) Is it true that $\mathcal{F}_1 \subset \mathcal{F}_2$? Let $\mathcal{F}_3 = \sigma(\{I_1, \dots, I_4, J_1, \dots, J_6\})$. Is it true that $\mathcal{F}_1 \subset \mathcal{F}_3$?

Solution: No, $\mathcal{F}_1 \not\subset \mathcal{F}_2$, since the atoms $\{I_1, \dots, I_4\} \notin \mathcal{F}_2$. Yes, $\mathcal{F}_1 \subset \mathcal{F}_3$, since $\{I_1, \dots, I_4\} \in \mathcal{F}_2$ as well as their countable unions.

Define the random variables $X = 1_{\{\omega \in I_1 \cup J_2\}}$, $Y = \omega 1_{\{\omega \in J_1 \cup I_2\}}$.

c) Is X $\mathcal{F}_1$-measurable? Is it $\mathcal{F}_2$-measurable and $\mathcal{F}_3$-measurable? What about Y?

Solution: We can write $X = 1_{\{\omega \in [0, \frac{1}{3})\}}$. Knowing that $\left[0, \frac{1}{3}\right) \notin \mathcal{F}_1$, and $\left[0, \frac{1}{3}\right) \in \mathcal{F}_2 \cap \mathcal{F}_3$, we conclude that X is $\mathcal{F}_2$-measurable and $\mathcal{F}_3$-measurable but not $\mathcal{F}_1$-measurable. Similarly, we can write $Y = \omega 1_{\{[0,\frac{1}{6}) \cup [\frac{1}{4}, \frac{1}{2})\}}$. Since Y takes a unique value for each $\omega \in \left\{\left[0, \frac{1}{6}\right) \cup \left[\frac{1}{4}, \frac{1}{2}\right)\right\}$, it is not $\mathcal{F}_1$, $\mathcal{F}_2$ or $\mathcal{F}_3$ measurable.

d) Calculate the expectations $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

Solution:

$$\mathbb{E}(X) = \mathbb{E}\left(1_{\{\omega \in [0, \frac{1}{3})\}}\right) = \mathbb{P}\left(\omega \in \left[0, \frac{1}{3}\right)\right) = \frac{1}{3}$$

and

$$\mathbb{E}(Y) = \mathbb{E}\left(\omega 1_{\{[0,\frac{1}{6}) \cup [\frac{1}{4}, \frac{1}{2})\}}\right) = \int_0^1 \omega \times 1_{\{[0,\frac{1}{6}) \cup [\frac{1}{4}, \frac{1}{2})\}} \, d\mathbb{P}(\omega)$$

$$= \int_0^{\frac{1}{6}} \omega \, d\mathbb{P}(\omega) + \int_{\frac{1}{4}}^{\frac{1}{2}} \omega \, d\mathbb{P}(\omega) = \frac{1}{2}\left(\left(\frac{1}{6}\right)^2 + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2\right) = \frac{31}{288}.$$

Define the random variable $Z = 1_{\{\omega \in J_2 \cup J_3 \cup J_4\}}$.

e) What is the covariance of X and Z? Are they independent?

Solution: We can write $Z = 1_{\{\omega \in [\frac{1}{6}, \frac{2}{3})\}}$. In addition, we can calculate $\mathbb{E}(Z) = \frac{1}{2}$. Then,

$$\mathrm{Cov}(X, Z) = \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z) = \mathbb{E}(1_{\{\omega \in [0, \frac{1}{3})\}} 1_{\{\omega \in [\frac{1}{6}, \frac{2}{3})\}}) - \mathbb{E}(X)\mathbb{E}(Z)$$

$$= \mathbb{E}(1_{\{\omega \in [0, \frac{1}{3}) \cap [\frac{1}{6}, \frac{2}{3})\}}) - \mathbb{E}(X)\mathbb{E}(Z) = \mathbb{E}(1_{\{\omega \in [\frac{1}{6}, \frac{1}{3})\}}) - \mathbb{E}(X)\mathbb{E}(Z) = \frac{1}{6} - \frac{1}{3}\frac{1}{2} = 0.$$

Since X and Z are uncorrelated Bernoulli random variables, we know from Exercise 1.e) that they are independent.

Exercise 3. (12 points) Consider an infinite sequence of coin flips, where the sample space is the set of all infinite binary sequences, $\Omega = \{0,1\}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, let A_n be the event that the first n coin flips are all heads. That is,

$$A_n = \{\omega = (\omega_1, \omega_2, \dots) \in \Omega : \omega_1 = \cdots = \omega_n = 1\}.$$

Let $\mathcal{F} = \sigma(\{A_n : n \in \mathbb{N}\})$.

a) Let B_n be the event that the first tail appears on the n-th flip, for $n \in \mathbb{N}$. That is,

$$B_n = \{\omega = (\omega_1, \omega_2, \dots) \in \Omega : \omega_1 = \cdots = \omega_{n-1} = 1, \omega_n = 0\}.$$

Is $B_n \in \mathcal{F}$?

Solution: B_n is the event that the first tail appears on the n-th flip, for $n \in \mathbb{N}$. We can write $B_n = A_{n-1} \setminus A_n$ (with $A_0 = \Omega$), which shows $B_n \in \mathcal{F}$.

b) What are the atoms of $\mathcal{F}$?

Solution: The collection $\mathcal{P} = \{B_n\}_{n\in\mathbb{N}} \cup \{A_\infty\}$ forms a countable partition of Ω. Any generator A_n can be written as a union of elements from this partition: $A_n = (\bigcup_{k=n+1}^{\infty} B_k) \cup A_\infty$. It follows that $\mathcal{F} = \sigma(\{A_n\}) = \sigma(\mathcal{P})$. The atoms of $\mathcal{F}$ are therefore the elements of this partition:

- The singleton event $A_\infty = \{(1,1,1,\dots)\}$.
- For each $n \in \mathbb{N}$, the event $B_n = \{\omega : \omega_1 = \cdots = \omega_{n-1} = 1, \omega_n = 0\}$.

c) Does $\mathcal{F}$ contain an event C_2, that the second flip is tails (i.e., $\omega_2 = 0$) ?

Solution: No, $\mathcal{F}$ does not contain the event C_2. An event in $\mathcal{F} = \sigma(\{B_n, n \in \mathbb{N}\} \cup \{A_\infty\})$ must be a countable union of the atoms $\{B_n\}_{n\in\mathbb{N}} \cup \{A_\infty\}$. This means that for any atom, such as B_1, an event $E \in \mathcal{F}$ must either contain all of B_1 or be disjoint from it.

Let's examine C_2 in relation to the atom $B_1 = \{\omega : \omega_1 = 0\}$.

- The outcome $\omega_a = (0,0,1,1,\dots)$ is in B_1 and is also in C_2.
- The outcome $\omega_b = (0,1,1,1,\dots)$ is in B_1 but is not in C_2.

Since C_2 contains some outcomes from the atom B_1 but not all of them, C_2 "splits" an atom of $\mathcal{F}$. Therefore, C_2 cannot be formed by a union of the atoms of $\mathcal{F}$ and thus $C_2 \notin \mathcal{F}$.

d) Let G_{ij} be the event that the first flip is i and the second flip is j. That is, $G_{00} = \{\omega \in \Omega \colon \omega_1 = 0, \omega_2 = 0\}$, $G_{01} = \{\omega \in \Omega \colon \omega_1 = 0, \omega_2 = 1\}$, and so on. Let $\mathcal{G} = \sigma(\{G_{00}, G_{01}, G_{10}, G_{11}\})$. Describe the σ-field $\mathcal{F} \cap \mathcal{G}$. That is, which sets is it generated by? How many elements does it have?

Hint: Recall that an intersection of two σ-fields is a σ-field.

Solution: The σ-field $\mathcal{F} \cap \mathcal{G}$ is given by $\sigma(\{G_{00} \cup G_{01}, G_{10}, G_{11}\})$. It has eight elements which are all possible unions of these three sets. To see why this is true, consider the following correspondence between the atoms of $\mathcal{F}$ and $\mathcal{G}$:

- $B_1 = G_{00} \cup G_{01}$
- $B_2 = G_{10}$
- $A_\infty \cup B_3 \cup B_4 \cdots = G_{11}$

Thus, we see that the set G_{10} is in both σ-fields. The sets G_{00} and G_{01} are not in $\mathcal{F}$, but their union is. Likewise, the sets A_∞, B_3, B_4, ... are not in $\mathcal{G}$, but their countable union is.