Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 5 – Solutions

Exercise 1:

(a) Let $f\colon X \to S$ be a map from a topological space X to a set S. Show that if X is connected and if f is *locally constant*, i.e., for every $x \in X$ there exists a neighborhood U of x in X such that $f|_U\colon U \to S$ is constant, then f is constant.

[Hint: Show that f is continuous when S is endowed with the discrete topology.]

(b) Let M and N be smooth manifolds and let $F\colon M \to N$ be a smooth map. Assume that M is connected. Show that $dF_p\colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant.

[Hint: Use part (a). You may also use (without proof) the fact that any topological manifold is locally (path) connected.]

Solution:

(a) We endow S with the discrete topology, and we claim that $f\colon X \to S$ is continuous. Since then the singletons in S are open, to prove the claim, it suffices to show that the fibers of f are open subsets of X. Fix $s \in S$ and pick $x \in f^{-1}(s)$. Since f is locally constant, there exists an open neighborhood U of x in X such that $f|_U\colon U \to S$ is constant, so for every $u \in U$ we have $f(u) = f(x) = s$, and hence $u \in f^{-1}(s)$. Therefore, the open neighborhood U of x is contained in the fiber $f^{-1}(s)$; in other words, $x \in U \subseteq f^{-1}(s)$. Since $x \in f^{-1}(s)$ was arbitrary, $f^{-1}(s)$ is an open subset of X, and since $s \in S$ was arbitrary, we conclude that f is continuous.

Since S is endowed with the discrete topology, every singleton in S is also closed, and thus every fiber of f is also closed, since f is continuous. In other words, the fibers of f are both closed and open subsets of X, which is a connected space by assumption, and hence each one of them is either empty or the whole space X. It follows that f is constant.

(b) Assume first that F is constant and let $p \in M$. For every $v \in T_pM$ and every $f \in C^\infty(N)$ we have $dF_p(v)(f) = v(f \circ F) = 0$ by *Lemma 3.5*(a), since the composite map $f \circ F\colon M \to \mathbb{R}$ is constant. In conclusion, dF_p is the zero linear transformation for every $p \in M$.

Assume now that dF_p is the zero map for each $p \in M$. By assumption and by (a), to prove that F is constant, it suffices to show that F is locally constant. Fix $p \in M$. Since F is smooth, there are smooth charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth. By shrinking U if necessary, we may assume that U is connected, and thus $\varphi(U)$ is also connected. Now, for each $q \in U$ we know that the differential dF_q is represented in coordinate bases by the Jacobian matrix of $\widehat{F}$. Since $dF_q = \mathbb{O}$ for every $q \in U$ by assumption, we infer that

$$\frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{q}) = 0 \quad \text{for every } i, \text{ every } j, \text{ and every } \widehat{q} = \varphi(q) \in \varphi(U).$$

Therefore, $\widehat{F}$ is constant on $\varphi(U)$, and hence $F = \varphi \circ \widehat{F} \circ \psi^{-1}$ is constant on U. Since $p \in M$ was arbitrary, we conclude that F is locally constant, as desired.

Exercise 2: Prove the following assertions:

(a) The quotient map $\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth.

(b) A map $F \colon \mathbb{RP}^n \to M$ to a smooth manifold M is smooth if and only if the composite map $F \circ \pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to M$ is smooth.

(c) For any point $p \in \mathbb{R}^{n+1} \setminus \{0\}$, the differential $d\pi_p \colon T_p\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \to T_{[p]}\mathbb{RP}^n$ is surjective (i.e., $\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is a smooth submersion) and its kernel is the subspace generated by p.

Solution:

(a) Note that the coordinate representation of π with respect to the smooth charts $\big(\pi^{-1}(U_i), \mathrm{Id}\big)$ for $\mathbb{R}^{n+1} \setminus \{0\}$ and (U_i, φ_i) for $\mathbb{RP}^n$ is

$$\begin{aligned} \widehat{\pi} \colon \mathbb{R}^{n+1}_{x_i \neq 0} &\to \mathbb{R}^n \\ (x_0, \dots, x_n) &\mapsto \frac{1}{x_i}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n). \end{aligned}$$

Since this map is clearly smooth and since the charts $\big(\pi^{-1}(U_i), \mathrm{Id}\big)$ cover $\mathbb{R}^{n+1} \setminus \{0\}$, we conclude that π is smooth.

(b) Let $F \colon \mathbb{RP}^n \to M$ be a map such that $F \circ \pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to M$ is smooth. Consider the map

$$\begin{aligned} \Phi_i \colon U_i &\to \mathbb{R}^{n+1}_{x_i \neq 0} \\ [x] &\mapsto \frac{1}{x_i} x. \end{aligned}$$

Note that Φ_i is well-defined. Furthermore, it is smooth, as its coordinate representation with respect to the global charts (U_i, φ_i) and $(\mathbb{R}^{n+1}_{x_i \neq 0}, \mathrm{Id})$ is given by the map

$$\begin{aligned} \mathbb{R}^n &\to \mathbb{R}^{n+1}_{x_i \neq 0} \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_i, 1, x_{i+1}, \dots, x_n). \end{aligned}$$

Finally, notice that $\pi \circ \Phi_i = \mathrm{Id}_{U_i}$, hence Φ_i is a smooth *section* of π. Now it is straightforward to conclude: to show that F is smooth, it suffices to show that $F|_{U_i}$ is smooth for all i. But then, as $(F \circ \pi)|_{\pi^{-1}(U_i)}$ is smooth, we deduce that

$$(F \circ \pi)|_{\pi^{-1}(U_i)} \circ \Phi_i = F|_{U_i}$$

is smooth as well.

The converse direction follows directly from the fact that a composition of smooth maps is smooth; see *Proposition 2.11* (d).

(c) From the solution to part (b), we know that for every $0 \leq i \leq n$ there exists a smooth map $\Phi_i \colon U_i \to \mathbb{R}^{n+1} \setminus \{0\}$ such that $\pi \circ \Phi_i = \iota_{U_i}$, where ι_{U_i} is the inclusion of U_i into $\mathbb{RP}^n$. Write $p = (p^1, \dots, p^{n+1})$, and for each $0 \leq i \leq n$ set $\widetilde{\Phi}_i = p^i \cdot \Phi_i$. Then we still have $\pi \circ \widetilde{\Phi}_i = \iota_{U_i}$, and moreover $\widetilde{\Phi}_i\big([p]\big) = p$. Hence,

$$d\pi_p \circ d(\widetilde{\Phi}_i)_{[p]} = d(\iota_{U_i})_{[p]},$$

and as the right hand side is an isomorphism, we infer that $d\pi_p$ is surjective. Therefore, π is a smooth submersion.

Let us show that $p \in \ker\big(d\pi_p\big)$. Let $f \in C^\infty(\mathbb{RP}^n)$ be arbitrary, and let $D_p\big|_p$ be the directional derivative at p with direction p defined in [*Exercise Sheet* 4, *Exercise* 2]. Then

$$d\pi_p\left(D_p\big|_p\right)(f) = D_p\big|_p (f \circ \pi) = \frac{d}{dt}\bigg|_{t=0} \big(f \circ \pi\big)(p + tp) = 0$$

as $t \mapsto \pi(p + tp)$ is constant. By [*Exercise Sheet* 4, *Exercise* 2], $D_p\big|_p$ corresponds to p under the natural identification $T_p\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \cong \mathbb{R}^{n+1}$. Thus, the kernel of $d\pi_p$ is generated by p.

Exercise 3:

(a) Prove the following assertions:

(i) A composition of smooth submersions is a smooth submersion.

(ii) A composition of smooth immersions is a smooth immersion.

(iii) A composition of smooth embeddings is a smooth embedding.

(b) Show by means of a counterexample that a composition of smooth maps of constant rank need not have constant rank.

(c) Let M and N be smooth manifolds and let $F \colon M \to N$ be a map. Prove the following assertions:

(i) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(ii) If $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Solution:

(a) First, we show (i). Let $F\colon M \to N$ and $G\colon N \to P$ be smooth submersions and fix $p \in M$. The composite map $G \circ F\colon M \to P$ is smooth by *Proposition 2.11*(d), and its differential at p is the linear map

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{(G\circ F)(p)}P$$

by *Proposition 3.7*(b), which is surjective, since both linear maps

$$dF_p\colon T_pM \to T_{F(p)}N \quad \text{and} \quad dG_{F(p)}\colon T_{F(p)} \to T_{(G\circ F)(p)}$$

are surjective by assumption. Since $p \in M$ was arbitrary, we conclude that $G \circ F$ is a smooth submersion.

Next, to prove (ii), we argue exactly as in (i), except that the word "surjective" is replaced by the word "injective".

Finally, we show (iii). Let $F\colon M \to N$ and $G\colon N \to P$ be smooth embeddings. By (ii) we know that the composite map $G \circ F\colon M \to P$ is a smooth immersion, so it remains to show that $G \circ F$ is a homeomorphism onto its image $(G \circ F)(M) \subseteq P$ in the subspace topology. To this end, note that F is a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology, and that G is a homeomorphism onto its image $G(N) \subseteq P$ in the subspace topology, so the restriction $G|_{F(M)}\colon F(M) \to G\big(F(M)\big)$ is also a homeomorphism. Therefore, the composite map $G \circ F$ is a homeomorphism onto its image $(G \circ F)(M) \subseteq P$ in the subspace topology, as required. In conclusion, $G \circ F$ is a smooth embedding.

(b) Consider the maps

$$\gamma\colon (0, 2\pi) \to \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$

and

$$\pi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto y.$$

Observe first that

$$\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$$

and

$$\|\gamma'(t)\| = \|(-\sin t, \cos t)\| = 1 \text{ for all } t \in (0, 2\pi),$$

so γ is an injective smooth immersion; see *Example 4.4*(1). Moreover, π is a surjective smooth submersion by *Exercise* 4(a). Hence, both γ and π are smooth maps of constant rank. However, the composite map

$$\pi \circ \gamma\colon (0, 2\pi) \to \mathbb{R},\ t \mapsto \sin t$$

does not have constant rank, because its derivative

$$(\pi \circ \gamma)'\colon (0, 2\pi) \to \mathbb{R},\ t \mapsto -\cos t$$

vanishes for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

(c) Recall that a local diffeomorphism is a smooth map by *Proposition 2.9*(a).

(c)(i) Assume first that F is a local diffeomorphism. According to *Proposition 3.6*(d), for any $p \in M$, the differential of F at p is an $\mathbb{R}$-linear isomorphism, and thus both injective and surjective. Hence, F is both a smooth immersion and a smooth submersion.

Assume now that F is both a smooth immersion and a smooth submersion. Then for every $p \in M$, its differential dF_p is both injective and surjective, and thus an $\mathbb{R}$-linear isomorphism. It follows from *Theorem 4.8* that F is a local diffeomorphism.

(c)(ii) Since $\dim M = \dim N$, for any $p \in M$, the differential $dF_p \colon T_pM \to T_{F(p)}N$ is an $\mathbb{R}$-linear map between $\mathbb{R}$-vector spaces of the same dimension. Thus, dF_p is injective or surjective if and only if it an isomorphism. Therefore, F is a smooth immersion if and only if F is a smooth submersion, and hence (ii) follows immediately from (i).

Exercise 4 (to be submitted):

(a) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Show that each of the projection maps

$$\pi_i \colon M_1 \times \ldots \times M_k \to M_i$$

is a smooth submersion.

(b) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Choosing arbitrarily points $p_1 \in M_1, \ldots, p_k \in M_k$, for each $1 \leq j \leq k$ consider the map

$$\iota_j \colon M_j \to M_1 \times \ldots \times M_k,\ x \mapsto (p_1, \ldots, p_{j-1}, x, p_{j+1}, \ldots, p_k).$$

Show that each ι_j is a smooth embedding.

(c) Show that the inclusion map $\iota \colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding, where $n \geq 1$.

(d) Show that the map

$$G \colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u, v) \mapsto \big((2 + \cos 2\pi u)\cos 2\pi v,\ (2 + \cos 2\pi u)\sin 2\pi v,\ \sin 2\pi u\big)$$

is a smooth immersion.

Solution:

(a) Fix $i \in \{1, \ldots, k\}$ and $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$. By part (a) of [*Exercise Sheet* 3, *Exercise* 4] we know that $\pi_i \colon M_1 \times \ldots \times M_k \to M_i$ is a smooth map, while by [*Exercise Sheet* 4, *Exercise* 3] we know that

$$\begin{aligned} T_p\big(M_1 \times \ldots \times M_k\big) &\longrightarrow T_{p_1}M_1 \oplus \ldots \oplus T_{p_i}M_i \oplus \ldots \oplus T_{p_k}M_k \\ v &\mapsto \big(d(\pi_1)_p(v), \ldots, d(\pi_i)_p(v), \ldots, d(\pi_k)_p(v)\big) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism. Using the above identification, we infer that the differential of π_i at p,

$$d(\pi_i)_p \colon T_{p_1}M_1 \oplus \ldots \oplus T_{p_i}M_i \oplus \ldots \oplus T_{p_k}M_k \to T_{p_i}M_i,$$

is surjective. Since $p \in M_1 \times \ldots \times M_k$ was arbitrary, we conclude that π_i is a smooth submersion.

(b) Fix $j \in \{1, \ldots, k\}$ and points $p_1 \in M_1, \ldots, p_{j-1} \in M_{j-1}, p_{j+1} \in M_{j+1}, \ldots, p_k \in M_k$. We have already seen in the solution of [*Exercise Sheet* 4, *Exercise* 3] that the map

$$\iota_j \colon M_j \to M_1 \times \ldots \times M_k,\ x \mapsto (p_1, \ldots, p_{j-1}, x, p_{j+1}, \ldots, p_k)$$

is smooth, and it is also clear that ι_j is a homeomorphism onto its image

$$\iota_j(M_j) = \{p_1\} \times \cdots \times \{p_{j-1}\} \times M_j \times \{p_{j+1}\} \times \cdots \times \{p_k\}.$$

Moreover, given a point $p_j \in M_j$, using the identification

$$T_p\big(M_1 \times \ldots \times M_k\big) \cong T_{p_1}M_1 \oplus \ldots \oplus T_{p_i}M_i \oplus \ldots \oplus T_{p_k}M_k,$$

where $p \coloneqq (p_1, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_k) \in M_1 \times \ldots M_k$, we infer that the differential of ι_j at p,

$$d(\iota_j)_{p_j} \colon T_{p_j}M_j \to T_{p_1}M_1 \oplus \ldots \oplus T_{p_j}M_j \oplus \ldots \oplus T_{p_k}M_k,$$

is injective. In conclusion, ι_j is a smooth embedding.

(c) Consider the graph coordinates $\big(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm}\big)$ for $\mathbb{S}^n$; see *Example 1.10*(2). We have shown in *Example 2.12* that the inclusion map $\iota \colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth, because its coordinate representation with respect to any of the graph coordinates is

$$\widehat{\iota}(u^1, \ldots, u^n) = \left(u^1, \ldots, u^{i-1}, \pm\sqrt{1 - \|u\|^2}, u^i, \ldots, u^n\right),$$

which is smooth on its domain, the unit ball $\mathbb{B}^n = \{u = (u^1, \ldots, u^n) \in \mathbb{R}^n \mid \|u\| < 1\}$. The Jacobian matrix of the coordinate representation $\widehat{\iota} = \iota \circ (\varphi_i^{\pm})^{-1}$ of ι with respect to the graph coordinates has the form

$$\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\
\frac{\mp u^1}{\sqrt{1-\|u\|^2}} & \frac{\mp u^2}{\sqrt{1-\|u\|^2}} & \ldots & \frac{\mp u^{i-1}}{\sqrt{1-\|u\|^2}} & \frac{\mp u^i}{\sqrt{1-\|u\|^2}} & \frac{\mp u^{i+1}}{\sqrt{1-\|u\|^2}} & \ldots & \frac{\mp u^{n-1}}{\sqrt{1-\|u\|^2}} & \frac{\mp u^n}{\sqrt{1-\|u\|^2}} \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}.$$

In particular, we observe that each of these $(n+1) \times n$ matrices (which represent the differential of ι in coordinate bases) has rank n. Hence, ι is an injective smooth immersion. Since $\mathbb{S}^n$ is compact, by *Proposition 4.6*(c) we conclude that ι is a smooth embedding.

(d) The map G with component functions (G^1, G^2, G^3) is clearly smooth with Jacobian matrix

$$J_G(u,v) = \begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix}$$

$$= \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}.$$

The 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \end{pmatrix}$$

of J_G has determinant

$$D_{12}(u,v) := -4\pi^2\big(2+\cos(2\pi u)\big)\sin(2\pi u),$$

the 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}$$

of J_G has determinant

$$D_{13}(u,v) := 4\pi^2\big(2+\cos(2\pi u)\big)\cos(2\pi u)\sin(2\pi v),$$

and the 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}$$

of J_G has determinant

$$D_{23}(u,v) := -4\pi^2\big(2+\cos(2\pi u)\big)\cos(2\pi u)\cos(2\pi v).$$

Observe now that for each $(u,v)\in\mathbb{R}^2$, at least one of the determinants $D_{12}(u,v)$, $D_{13}(u,v)$ and $D_{23}(u,v)$ is non-zero, since $\cos(2\pi\theta)$ and $\sin(2\pi\theta)$ do not vanish simultaneously. This implies[1] that $\operatorname{rk}\big(J_G(u,v)\big) = 2$ for all $(u,v)\in\mathbb{R}^2$. In conclusion, G is a smooth immersion, as claimed.

[1] By linear algebra we know that an $(m\times n)$-matrix with $m<n$ has full rank if and only if it has an invertible $(m\times m)$-submatrix.

Exercise 5: Consider the map

$$F\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (2 + \tanh t) \cdot (\cos t,\, \sin t).$$

(a) Show that F is an injective smooth immersion.

(b) Show that F is a smooth embedding.
[Hint: Show that $F\colon \mathbb{R} \to U = \{x \in \mathbb{R}^2 \mid 1 < \|x\| < 3\}$ is a proper map.]

Solution:

(a) Clearly, F is smooth. Recall also that the function

$$t \in \mathbb{R} \mapsto \|F(t)\| = 2 + \tanh t$$

is strictly increasing, which implies that F is injective. Finally, to show that F is a smooth immersion, it suffices to show that $F'(t) \neq 0$ for every $t \in \mathbb{R}$. To this end, recall that

$$\frac{d}{dt}\tanh t = \frac{1}{\cosh^2 t}\,,\quad t \in \mathbb{R},$$

so we have

$$F'(t) = \left(-(2+\tanh t)\sin t + \frac{1}{\cosh^2 t}\cos t,\ (2+\tanh t)\cos t + \frac{1}{\cosh^2 t}\sin t\right),\quad t \in \mathbb{R},$$

and thus

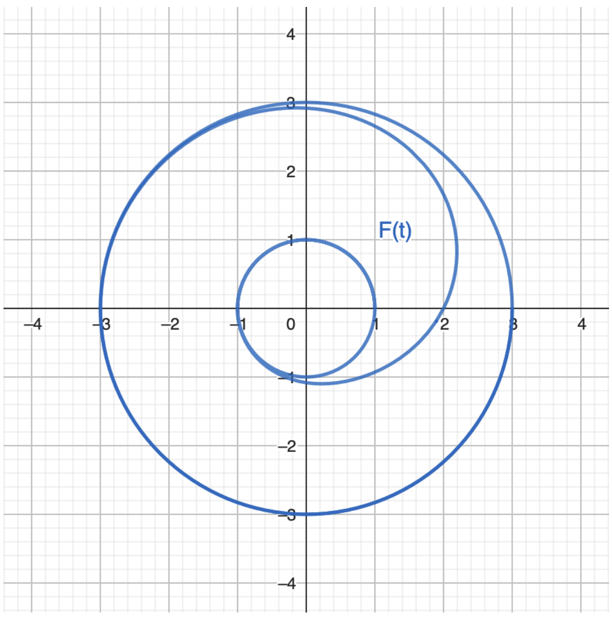
$$\|F'(t)\|^2 = (2+\tanh t)^2 + \frac{1}{\cosh^4 t} > 0 \quad \text{for all } t \in \mathbb{R},$$

which implies that $F'(t) \neq 0$ for every $t \in \mathbb{R}$, as desired.

(b) Consider the open annulus

$$U := \left\{x \in \mathbb{R}^2 \mid 1 < \|x\| < 3\right\} \subseteq \mathbb{R}^2$$

and note that $F(t) \in U$ for every $t \in \mathbb{R}$. (Incidentally, the image of $F|_{[-4\pi,4\pi]}$ has been plotted below.)



Thus, F may be viewed as an injective smooth immersion $F\colon \mathbb{R} \to U$. Since the inclusion map $\iota\colon U \hookrightarrow \mathbb{R}^2$ is a smooth embedding by *Example 4.4*(3), in view of *Exercise* 2(a)(iii) and *Proposition 4.6*(b), to prove (b), it suffices to show that $F\colon \mathbb{R} \to U$ is a *proper* map; in other words, given a compact subset K of U, we have to show that $F^{-1}(K)$ is a compact subset of $\mathbb{R}$, or equivalently that it is closed and bounded. Since $K \subseteq U$ is compact and $U \subseteq \mathbb{R}^2$ is Hausdorff, K is a closed subset of U, and since F is continuous, $F^{-1}(K)$ is a closed subset of $\mathbb{R}$. Now, denote by m (resp. M) the minimum (resp. the maximum) norm of the points of K, and observe that $[m, M] \subseteq (1, 3)$. Denote also by ℓ (resp. L) the preimage of m (resp. M) under the strictly increasing function

$$g\colon \mathbb{R} \to (1,3),\ t \mapsto \|F(t)\| = 2 + \tanh t$$

and note that $F^{-1}(K) \subseteq [\ell, L]$, which shows that $F^{-1}(K)$ is a bounded subset of $\mathbb{R}$. This finishes the proof of (b).