



Differential Geometry II - Smooth Manifolds

Winter Term 2026/2027

Lecturer: Dimitri Stelio Wyss

Assistant: Eric Yen-Yo Chen

Exercise Sheet 5

Exercise 1: Prove the following assertions:

- (a) *Tangent vectors as velocity vectors of smooth curves:* Let M be a smooth manifold. If $p \in M$, then for any $v \in T_p M$ there exists a smooth curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.
- (b) *The velocity of a composite curve:* If $F: M \rightarrow N$ is a smooth map and if $\gamma: J \rightarrow M$ is a smooth curve, then for any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma: J \rightarrow N$ is given by

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$

- (c) *Computing the differential using a velocity vector:* If $F: M \rightarrow N$ is a smooth map, $p \in M$ and $v \in T_p M$, then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma: J \rightarrow M$ such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

Exercise 2:

- (a) Let $f: X \rightarrow S$ be a map from a topological space X to a set S . Show that if X is connected and if f is *locally constant*, i.e., for every $x \in X$ there exists a neighborhood U of x in X such that $f|_U: U \rightarrow S$ is constant, then f is constant.

[Hint: Show that f is continuous when S is endowed with the discrete topology.]

- (b) Let M and N be smooth manifolds and let $F: M \rightarrow N$ be a smooth map. Assume that M is connected. Show that $dF_p: T_p M \rightarrow T_{F(p)} N$ is the zero map for each $p \in M$ if and only if F is constant.

[Hint: Use part (a). You may also use (without proof) the fact that any topological manifold is locally (path) connected.]

Exercise 3:

Prove the following assertions:

- (a) The quotient map $\pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$ is smooth.
- (b) A map $F: \mathbb{RP}^n \rightarrow M$ to a smooth manifold M is smooth if and only if the composite map $F \circ \pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow M$ is smooth.
- (c) For any point $p \in \mathbb{R}^{n+1} \setminus \{0\}$, the differential $d\pi_p: T_p(\mathbb{R}^{n+1} \setminus \{0\}) \rightarrow T_p\mathbb{RP}^n$ is surjective (i.e., $\pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$ is a smooth submersion) and its kernel is the subspace generated by p .

Exercise 4:

Definition. A Lie group is a smooth manifold G that has a group structure such that the multiplication map $m: G \times G \rightarrow G$ and the inversion map $i: G \rightarrow G$ are both smooth.

- (a) Show that S^1 and $GL_n(\mathbb{R})$ are Lie groups.
- (b) Under the identification $T_e G \times G = T_e G \oplus T_e G$ of last week's exercise 2, show that $dm_e(X, Y) = X + Y$ for all $X, Y \in T_e G$.
- (c) Show that $di_e(X) = -X$.
- (d) Show that the tangent bundle of any Lie group is trivial.
- (e) (Optional) Read about the hairy ball theorem and use it to show that the tangent bundle of S^2 is non-trivial.