Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 5

Exercise 1:

(a) Let $f\colon X \to S$ be a map from a topological space X to a set S. Show that if X is connected and if f is *locally constant*, i.e., for every $x \in X$ there exists a neighborhood U of x in X such that $f|_U\colon U \to S$ is constant, then f is constant.

[Hint: Show that f is continuous when S is endowed with the discrete topology.]

(b) Let M and N be smooth manifolds and let $F\colon M \to N$ be a smooth map. Assume that M is connected. Show that $dF_p\colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant.

[Hint: Use part (a). You may also use (without proof) the fact that any topological manifold is locally (path) connected.]

Exercise 2:
Prove the following assertions:

(a) The quotient map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth.

(b) A map $F\colon \mathbb{RP}^n \to M$ to a smooth manifold M is smooth if and only if the composite map $F \circ \pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to M$ is smooth.

(c) For any point $p \in \mathbb{R}^{n+1} \setminus \{0\}$, the differential $d\pi_p\colon T_p\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \to T_{[p]}\mathbb{RP}^n$ is surjective (i.e., $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is a smooth submersion) and its kernel is the subspace generated by p.

Exercise 3:

(a) Prove the following assertions:

  (i) A composition of smooth submersions is a smooth submersion.

  (ii) A composition of smooth immersions is a smooth immersion.

  (iii) A composition of smooth embeddings is a smooth embedding.

(b) Show by means of a counterexample that a composition of smooth maps of constant rank need not have constant rank.

(c) Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Prove the following assertions:

  (i) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

  (ii) If $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Exercise 4 (to be submitted by Thursday, 16.10.2025, 16:00):

(a) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Show that each of the projection maps

$$\pi_i\colon M_1 \times \ldots \times M_k \to M_i$$

is a smooth submersion.

(b) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Choosing arbitrarily points $p_1 \in M_1, \ldots, p_k \in M_k$, for each $1 \leq j \leq k$ consider the map

$$\iota_j\colon M_j \to M_1 \times \ldots \times M_k,\ x \mapsto (p_1, \ldots, p_{j-1}, x, p_{j+1}, \ldots, p_k).$$

Show that each ι_j is a smooth embedding.

(c) Show that the inclusion map $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding, where $n \geq 1$.

(d) Show that the map

$$G\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \big((2+\cos 2\pi u)\cos 2\pi v,\ (2+\cos 2\pi u)\sin 2\pi v,\ \sin 2\pi u\big)$$

is a smooth immersion.

Exercise 5:
Consider the map

$$F\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (2+\tanh t)\cdot(\cos t,\ \sin t).$$

(a) Show that F is an injective smooth immersion.

(b) Show that F is a smooth embedding.
[Hint: Show that $F\colon \mathbb{R} \to U = \{x \in \mathbb{R}^2 \mid 1 < \|x\| < 3\}$ is a proper map.]