



Differential Geometry II - Smooth Manifolds

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Exercise Sheet 4 – Solutions

Exercise 1: Let M , N and P be smooth manifolds, let $F: M \rightarrow N$ and $G: N \rightarrow P$ be smooth maps, and let $p \in M$. Prove the following assertions:

- (a) The map $dF_p: T_pM \rightarrow T_{F(p)}N$ is $\mathbb{R}$ -linear.
- (b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p: T_pM \rightarrow T_{(G \circ F)(p)}P$.
- (c) $d(\text{Id}_M)_p = \text{Id}_{T_pM}: T_pM \rightarrow T_pM$.
- (d) If F is a diffeomorphism, then $dF_p: T_pM \rightarrow T_{F(p)}N$ is an isomorphism, and it holds that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Solution: Recall that T_pM is the vector space of derivations at p , i.e. $\mathbb{R}$ -linear maps $v: C^\infty(M) \rightarrow \mathbb{R}$ with $v(fg) = f(p)v(g) + g(p)v(f)$, and that for $v \in T_pM$ the differential is defined by

$$dF_p(v)(f) = v(f \circ F), \quad f \in C^\infty(N).$$

This is well defined: $f \circ F \in C^\infty(M)$ since F is smooth, and $dF_p(v)$ is a derivation at $F(p)$. Indeed, $f \mapsto f \circ F$ is linear, so $dF_p(v)$ is linear, and for $f, g \in C^\infty(N)$,

$$\begin{aligned} dF_p(v)(fg) &= v((f \circ F)(g \circ F)) \\ &= f(F(p))v(g \circ F) + g(F(p))v(f \circ F) \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f). \end{aligned}$$

- (a) Let $v, w \in T_pM$, $a, b \in \mathbb{R}$ and $f \in C^\infty(N)$. Since the vector space operations on T_pM are defined pointwise,

$$dF_p(av + bw)(f) = (av + bw)(f \circ F) = av(f \circ F) + bw(f \circ F) = (a dF_p(v) + b dF_p(w))(f).$$

As f was arbitrary, $dF_p(av + bw) = a dF_p(v) + b dF_p(w)$.

- (b) $G \circ F$ is smooth, so $d(G \circ F)_p$ is defined. For $v \in T_pM$ and $f \in C^\infty(P)$ we have $f \circ G \in C^\infty(N)$ and

$$d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = dF_p(v)(f \circ G) = dG_{F(p)}(dF_p(v))(f).$$

Hence $d(G \circ F)_p(v) = dG_{F(p)}(dF_p(v))$ for all $v \in T_pM$.

- (c) For $v \in T_p M$ and $f \in C^\infty(M)$: $d(\text{Id}_M)_p(v)(f) = v(f \circ \text{Id}_M) = v(f)$. So $d(\text{Id}_M)_p(v) = v$.
- (d) Since F is a diffeomorphism, $F^{-1}: N \rightarrow M$ is smooth, and $F^{-1} \circ F = \text{Id}_M$, $F \circ F^{-1} = \text{Id}_N$. Applying (b) and (c) at p and at $F(p)$ respectively (note $F^{-1}(F(p)) = p$):

$$\begin{aligned} d(F^{-1})_{F(p)} \circ dF_p &= d(F^{-1} \circ F)_p = d(\text{Id}_M)_p = \text{Id}_{T_p M}, \\ dF_p \circ d(F^{-1})_{F(p)} &= d(F \circ F^{-1})_{F(p)} = d(\text{Id}_N)_{F(p)} = \text{Id}_{T_{F(p)} N}. \end{aligned}$$

Thus dF_p is bijective with inverse $d(F^{-1})_{F(p)}$, and it is linear by (a), hence a linear isomorphism with $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Exercise 2 (*The tangent space to a product manifold*):

Let $M_1, \dots, M_k$ be smooth manifolds, where $k \geq 2$. For each $j \in \{1, \dots, k\}$, let

$$\pi_j: M_1 \times \dots \times M_k \rightarrow M_j$$

be the projection onto the j -th factor M_j . Show that for any point $p = (p_1, \dots, p_k) \in M_1 \times \dots \times M_k$, the map

$$\begin{aligned} \alpha: T_p(M_1 \times \dots \times M_k) &\longrightarrow T_{p_1} M_1 \oplus \dots \oplus T_{p_k} M_k \\ v &\mapsto (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v)) \end{aligned}$$

is an $\mathbb{R}$ -linear isomorphism.

Solution: Write $M := M_1 \times \dots \times M_k$ and $n_j := \dim M_j$. Each π_j is smooth, and each component $v \mapsto d(\pi_j)_p(v)$ of α is linear by Exercise 1(a); hence α is $\mathbb{R}$ -linear.

Preliminary fact. If $G: X \rightarrow Y$ is a constant map between smooth manifolds, then $dG_x = 0$ for all $x \in X$. Indeed, every derivation $w \in T_x X$ vanishes on constants: $w(1) = w(1 \cdot 1) = 2w(1)$, so $w(1) = 0$, and by linearity $w(c) = 0$ for every constant c . Hence for $f \in C^\infty(Y)$, $dG_x(w)(f) = w(f \circ G) = 0$ since $f \circ G$ is constant.

Surjectivity. For $j \in \{1, \dots, k\}$ define

$$\iota_j: M_j \rightarrow M, \quad \iota_j(x) = (p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_k).$$

Each component of ι_j is either constant or the identity, so ι_j is smooth, and $\iota_j(p_j) = p$. Moreover

$$\pi_j \circ \iota_j = \text{Id}_{M_j}, \quad \pi_i \circ \iota_j \equiv p_i \text{ (constant) for } i \neq j.$$

Define the linear map

$$\beta: T_{p_1} M_1 \oplus \dots \oplus T_{p_k} M_k \rightarrow T_p M, \quad \beta(v_1, \dots, v_k) = \sum_{j=1}^k d(\iota_j)_{p_j}(v_j).$$

By Exercise 1(b),(c) and the preliminary fact, the i -th component of $\alpha(\beta(v_1, \dots, v_k))$ is

$$d(\pi_i)_p \left(\sum_j d(\iota_j)_{p_j}(v_j) \right) = \sum_j d(\pi_i \circ \iota_j)_{p_j}(v_j) = d(\text{Id}_{M_i})_{p_i}(v_i) + \sum_{j \neq i} 0 = v_i.$$

Hence $\alpha \circ \beta = \text{Id}$, so α is surjective.

Conclusion. Since the tangent space of an n -manifold at any point has dimension n , and M is a smooth manifold of dimension $n_1 + \dots + n_k$,

$$\dim T_p M = n_1 + \dots + n_k = \sum_{j=1}^k \dim T_{p_j} M_j = \dim (T_{p_1} M_1 \oplus \dots \oplus T_{p_k} M_k).$$

A surjective linear map between finite-dimensional vector spaces of the same dimension is injective by rank-nullity. Therefore α is an $\mathbb{R}$ -linear isomorphism (with inverse β).

Exercise 3: Consider the inclusion map $\iota: \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, where both $\mathbb{S}^2$ and $\mathbb{R}^3$ are endowed with the standard smooth structure. Let $p = (p^1, p^2, p^3) \in \mathbb{S}^2$ with $p^3 > 0$. What is the image of the differential $d\iota_p: T_p \mathbb{S}^2 \rightarrow T_p \mathbb{R}^3$?

Solution: Let (x^1, x^2, x^3) be standard coordinates on $\mathbb{R}^3$, so that $T_p \mathbb{R}^3$ has basis

$$\partial/\partial x^1|_p, \partial/\partial x^2|_p, \partial/\partial x^3|_p.$$

We claim

$$\text{im}(d\iota_p) = \left\{ v^1 \frac{\partial}{\partial x^1} \Big|_p + v^2 \frac{\partial}{\partial x^2} \Big|_p + v^3 \frac{\partial}{\partial x^3} \Big|_p \mid p^1 v^1 + p^2 v^2 + p^3 v^3 = 0 \right\},$$

i.e. under the identification $T_p \mathbb{R}^3 \cong \mathbb{R}^3$, the image is the plane $p^\perp$ orthogonal to p (the “geometric” tangent plane of the sphere at p).

Since $p^3 > 0$, p lies in the upper hemisphere $U = \{x \in \mathbb{S}^2 : x^3 > 0\}$, on which we have the graph chart $\varphi: U \rightarrow B := \{u \in \mathbb{R}^2 : |u| < 1\}$, $\varphi(x) = (x^1, x^2)$, with smooth inverse

$$\varphi^{-1}(u^1, u^2) = (u^1, u^2, \sqrt{1 - (u^1)^2 - (u^2)^2}).$$

Identifying $T_p U = T_p \mathbb{S}^2$ (U is open), the vectors $\partial/\partial u^i|_p := d(\varphi^{-1})_{\varphi(p)}(\partial/\partial u^i|_{\varphi(p)})$, $i = 1, 2$, form a basis of $T_p \mathbb{S}^2$ by Exercise 1(d). By Exercise 1(b),

$$d\iota_p \left(\frac{\partial}{\partial u^i} \Big|_p \right) = d(\iota \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial u^i} \Big|_{\varphi(p)} \right) = \sum_{l=1}^3 \frac{\partial(\iota \circ \varphi^{-1})^l}{\partial u^i}(\varphi(p)) \frac{\partial}{\partial x^l} \Big|_p,$$

where we used that the differential of a smooth map between open subsets of Euclidean spaces is given by its Jacobian in standard coordinates. With $\varphi(p) = (p^1, p^2)$ and $\sqrt{1 - (p^1)^2 - (p^2)^2} = p^3$, we get

$$d\iota_p \left(\frac{\partial}{\partial u^1} \Big|_p \right) = \frac{\partial}{\partial x^1} \Big|_p - \frac{p^1}{p^3} \frac{\partial}{\partial x^3} \Big|_p, \quad d\iota_p \left(\frac{\partial}{\partial u^2} \Big|_p \right) = \frac{\partial}{\partial x^2} \Big|_p - \frac{p^2}{p^3} \frac{\partial}{\partial x^3} \Big|_p.$$

These two vectors, with coefficient vectors $(1, 0, -p^1/p^3)$ and $(0, 1, -p^2/p^3)$, are linearly independent, so $d\iota_p$ is injective and its image is 2-dimensional. Both satisfy the equation $p^1 v^1 + p^2 v^2 + p^3 v^3 = 0$:

$$p^1 \cdot 1 + p^3 \cdot \left(-\frac{p^1}{p^3} \right) = 0, \quad p^2 \cdot 1 + p^3 \cdot \left(-\frac{p^2}{p^3} \right) = 0.$$

Since $p \neq 0$, the solution space of this equation is 2-dimensional, so it coincides with $\text{im}(d\iota_p)$, proving the claim.

Exercise 4 (Theorem C.15): Let $f \in C^{k+1}(\mathbb{R}^n)$ for some $k \geq 0$, and pick $a \in \mathbb{R}^n$ an arbitrary point. Show that for any $x \in \mathbb{R}^n$,

$$f(x) = P_k(x) + R_k(x)$$

where P_k is the k th order Taylor polynomial of f at a defined by

$$P_k(x) = f(a) + \sum_{m=1}^k \frac{1}{m!} \sum_{|I|=m} \partial_I f(a) (x-a)^I,$$

and R_k is the k th remainder term, given by

$$R_k(x) = \frac{1}{k!} \sum_{|I|=k+1} (x-a)^I \int_0^1 (1-t)^k \partial_I f(a + t(x-a)) dt.$$

Solution: *Conventions.* The sums run over ordered multi-indices $I = (i_1, \dots, i_m) \in \{1, \dots, n\}^m$, $|I| = m$, with

$$\partial_I f = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_m} f, \quad (x-a)^I = (x^{i_1} - a^{i_1})(x^{i_2} - a^{i_2}) \cdots (x^{i_m} - a^{i_m}).$$

In the Taylor polynomial the derivatives are evaluated at the base point, i.e. the summands are $\partial_I f(a)(x-a)^I$; this is what we prove. (With $\partial_I f(x)$ the identity fails: for $n = 1$, $f(x) = x^2$, $a = 0$, $k = 1$ one would get $P_1(x) = 2x^2$, $R_1(x) = x^2$.) Thus

$$P_k(x) = f(a) + \sum_{m=1}^k \frac{1}{m!} \sum_{|I|=m} \partial_I f(a) (x-a)^I.$$

Fix $x, a \in \mathbb{R}^n$ and write $\gamma(t) = a + t(x-a)$, $t \in [0, 1]$. We argue by induction on k .

Base case $k = 0$. Let $f \in C^1(\mathbb{R}^n)$. Then $t \mapsto f(\gamma(t))$ is C^1 with derivative $\sum_i \partial_i f(\gamma(t)) (x^i - a^i)$ by the chain rule, so by the fundamental theorem of calculus

$$f(x) - f(a) = \int_0^1 \frac{d}{dt} f(\gamma(t)) dt = \sum_{i=1}^n (x^i - a^i) \int_0^1 \partial_i f(\gamma(t)) dt.$$

This is exactly $f(x) = P_0(x) + R_0(x)$, since $P_0 = f(a)$, $0! = 1$ and $(1-t)^0 = 1$.

Induction step. Let $k \geq 1$, assume the statement holds for $k-1$ (for all C^k functions), and let $f \in C^{k+1}(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$. By the induction hypothesis

$$f(x) = P_{k-1}(x) + \frac{1}{(k-1)!} \sum_{|I|=k} (x-a)^I \int_0^1 (1-t)^{k-1} \partial_I f(\gamma(t)) dt. \quad (*)$$

For $|I| = k$ the function $\partial_I f$ is C^1 , hence $g_I(t) := \partial_I f(\gamma(t))$ is C^1 on $[0, 1]$ with $g'_I(t) = \sum_{j=1}^n (x^j - a^j) \partial_j \partial_I f(\gamma(t))$. Integrating by parts with the antiderivative $-\frac{(1-t)^k}{k}$ of $(1-t)^{k-1}$:

$$\begin{aligned} \int_0^1 (1-t)^{k-1} g_I(t) dt &= \left[-\frac{(1-t)^k}{k} g_I(t) \right]_0^1 + \frac{1}{k} \int_0^1 (1-t)^k g'_I(t) dt \\ &= \frac{1}{k} \partial_I f(a) + \frac{1}{k} \sum_{j=1}^n (x^j - a^j) \int_0^1 (1-t)^k \partial_j \partial_I f(\gamma(t)) dt. \end{aligned}$$

Inserting this into (*) and using $(k-1)! \cdot k = k!$ gives

$$f(x) = P_{k-1}(x) + \frac{1}{k!} \sum_{|I|=k} \partial_I f(a) (x-a)^I + \frac{1}{k!} \sum_{|I|=k} \sum_{j=1}^n (x^j - a^j) (x-a)^I \int_0^1 (1-t)^k \partial_j \partial_I f(\gamma(t)) dt.$$

The first two terms together are exactly $P_k(x)$ (the second is the $m = k$ summand). For the last term, note that $(j, I) \mapsto J := (j, i_1, \dots, i_k)$ is a bijection from $\{1, \dots, n\} \times \{I : |I| = k\}$ onto $\{J : |J| = k+1\}$, and by our conventions

$$(x^j - a^j)(x-a)^I = (x-a)^J, \quad \partial_j \partial_I f = \partial_J f.$$

Hence the last term equals

$$\frac{1}{k!} \sum_{|J|=k+1} (x-a)^J \int_0^1 (1-t)^k \partial_J f(a + t(x-a)) dt = R_k(x).$$

Therefore $f(x) = P_k(x) + R_k(x)$, which completes the induction. □