



Differential Geometry II - Smooth Manifolds

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Exercise Sheet 4

Exercise 1: Let M , N and P be smooth manifolds, let $F: M \rightarrow N$ and $G: N \rightarrow P$ be smooth maps, and let $p \in M$. Prove the following assertions:

- (a) The map $dF_p: T_p M \rightarrow T_{F(p)} N$ is $\mathbb{R}$ -linear.
- (b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p: T_p M \rightarrow T_{(G \circ F)(p)} P$.
- (c) $d(\text{Id}_M)_p = \text{Id}_{T_p M}: T_p M \rightarrow T_p M$.
- (d) If F is a diffeomorphism, then $dF_p: T_p M \rightarrow T_{F(p)} N$ is an isomorphism, and it holds that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Exercise 2 (*The tangent space to a product manifold*):

Let $M_1, \dots, M_k$ be smooth manifolds, where $k \geq 2$. For each $j \in \{1, \dots, k\}$, let

$$\pi_j: M_1 \times \dots \times M_k \rightarrow M_j$$

be the projection onto the j -th factor M_j . Show that for any point $p = (p_1, \dots, p_k) \in M_1 \times \dots \times M_k$, the map

$$\begin{aligned} \alpha: T_p(M_1 \times \dots \times M_k) &\longrightarrow T_{p_1} M_1 \oplus \dots \oplus T_{p_k} M_k \\ v &\mapsto (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v)) \end{aligned}$$

is an $\mathbb{R}$ -linear isomorphism.

Exercise 3: Consider the inclusion map $\iota: \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, where both $\mathbb{S}^2$ and $\mathbb{R}^3$ are endowed with the standard smooth structure. Let $p = (p^1, p^2, p^3) \in \mathbb{S}^2$ with $p^3 > 0$. What is the image of the differential $d\iota_p: T_p \mathbb{S}^2 \rightarrow T_p \mathbb{R}^3$?

Exercise 4 (Theorem C.15): Let $f \in C^{k+1}(\mathbb{R}^n)$ for some $k \geq 0$, and pick $a \in \mathbb{R}^n$ an arbitrary point. Show that for any $x \in \mathbb{R}^n$,

$$f(x) = P_k(x) + R_k(x)$$

where P_k is the k th order Taylor polynomial of f at a defined by

$$P_k(x) = f(a) + \sum_{m=1}^k \frac{1}{m!} \sum_{I:|I|=m} \partial_I f(x) (x-a)^I,$$

and R_k is the k th remainder term, given by

$$R_k(x) = \frac{1}{k!} \sum_{I:|I|=k+1} (x-a)^I \int_0^1 (1-t)^k \partial_I f(a+t(x-a)) dt.$$