Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 4

Exercise 1: Let M, N and P be smooth manifolds, let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps, and let $p \in M$. Prove the following assertions:

(a) The map $dF_p\colon T_pM \to T_{F(p)}N$ is $\mathbb{R}$-linear.

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{(G\circ F)(p)}P$.

(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}\colon T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $dF_p\colon T_pM \to T_{F(p)}N$ is an isomorphism, and it holds that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Exercise 2 (*The tangent space to a vector space*)**:**
Let V be a finite-dimensional $\mathbb{R}$-vector space with its standard smooth manifold structure. Fix a point $a \in V$.

(a) For each $v \in V$ define a map

$$D_v\big|_a\colon C^\infty(V) \longrightarrow \mathbb{R},\ f \mapsto \frac{d}{dt}\bigg|_{t=0} f(a+tv).$$

Show that $D_v\big|_a$ is a derivation at a.

(b) Show that the map

$$V \to T_aV,\ v \mapsto D_v\big|_a$$

is a canonical isomorphism, such that for any linear map $L\colon V \to W$ the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\cong"] \arrow[d, "L"'] & T_aV \arrow[d, "dL_a"] \\ W \arrow[r, "\cong"] & T_{La}W. \end{tikzcd}$$

Exercise 3 (*The tangent space to a product manifold*)**:**
Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. For each $j \in \{1, \ldots, k\}$, let

$$\pi_j \colon M_1 \times \ldots \times M_k \to M_j$$

be the projection onto the j-th factor M_j. Show that for any point $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$, the map

$$\begin{aligned} \alpha \colon T_p\big(M_1 \times \ldots \times M_k\big) &\longrightarrow T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k \\ v &\mapsto \big(d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v)\big) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism.

Exercise 4 (**to be submitted by Thursday, 09.10.2025, 16:00**)**:**
Consider the inclusion map $\iota \colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, where both $\mathbb{S}^2$ and $\mathbb{R}^3$ are endowed with the standard smooth structure. Let $p = (p^1, p^2, p^3) \in \mathbb{S}^2$ with $p^3 > 0$. What is the image of the differential $d\iota_p \colon T_p\mathbb{S}^2 \to T_p\mathbb{R}^3$?

Exercise 5:
Prove the following assertions:

(a) *Tangent vectors as velocity vectors of smooth curves*: Let M be a smooth manifold. If $p \in M$, then for any $v \in T_pM$ there exists a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

(b) *The velocity of a composite curve*: If $F \colon M \to N$ is a smooth map and if $\gamma \colon J \to M$ is a smooth curve, then for any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma \colon J \to N$ is given by

$$(F \circ \gamma)'(t_0) = dF\big(\gamma'(t_0)\big).$$

(c) *Computing the differential using a velocity vector*: If $F \colon M \to N$ is a smooth map, $p \in M$ and $v \in T_pM$, then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma \colon J \to M$ such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.