Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 3 – Solutions

Exercise 1 (*Equivalent characterizations of smoothness*)**:** Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Show that F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$ there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\big\{(U_\alpha, \varphi_\alpha)\big\}$ and $\big\{(V_\beta, \psi_\beta)\big\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha\big(U_\alpha \cap F^{-1}(V_\beta)\big)$ to $\psi_\beta(V_\beta)$.

Solution:

(a) We prove the two directions:

($\Rightarrow$) Suppose F is smooth and let $p \in M$. Then there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and such that $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$. Then $U \cap F^{-1}(V) = U$, and thus the charts (U, φ) and (V, ψ) satisfy the conditions specified in (a).

($\Leftarrow$) Assume that (a) holds and let $p \in M$. Let (U, φ) resp. (V, ψ) be the charts given by (a). Then, setting $U' := U \cap F^{-1}(V)$ and $\varphi' := \varphi|_{U'}$, we infer that (U', φ') is a smooth chart containing p such that $F(U') \subseteq V$ and such that $\psi \circ F \circ (\varphi')^{-1} \colon \varphi'(U') \to \psi(V)$ is smooth.

(b) We prove the two directions:

($\Rightarrow$) Suppose that F is smooth. By *Proposition 2.5*, F is continuous. Now, let (U, φ) and (V, ψ) be any smooth chart for M and N, respectively. We would like to show that the map $\widehat{F} := \psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$. If $U \cap F^{-1}(V)$ is empty, then there is nothing to prove. Otherwise, let $p \in U \cap F^{-1}(V)$ be arbitrary. By smoothness of F, there exist charts (W, η)

containing p and (Z,θ) containing $F(p)$ such that $F(W) \subseteq Z$ and such that $\theta \circ F \circ \eta^{-1}$ is smooth from $\eta(W)$ to $\theta(Z)$. In particular, we have

$$\widehat{F} = \psi \circ (\theta^{-1} \circ \theta) \circ F \circ (\eta^{-1} \circ \eta) \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ F \circ \eta^{-1}) \circ (\eta \circ \varphi^{-1})$$

on the open neighborhood $\varphi\big(U \cap W \cap F^{-1}(V)\big)$ containing $\varphi(p)$. As this is a composition of smooth functions between open subsets of Euclidean spaces, it follows that the function $\widehat{F}$ is smooth in a neighborhood of $\varphi(p)$. As $p \in U \cap F^{-1}(V)$ was arbitrary, we conclude that $\widehat{F}$ is smooth. Hence, the maximal smooth atlases of M and N satisfy (b).

($\Leftarrow$) Let $p \in M$. Let $(U_\alpha, \varphi_\alpha)$ be a smooth chart containing p and let (V_β, ψ_β) be a smooth chart containing $F(p)$. By hypothesis, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth from $\varphi_\alpha\big(U_\alpha \cap F^{-1}(V_\beta)\big)$ to $\psi_\beta(V_\beta)$. As $p \in M$ was arbitrary and since F is continuous, we infer that (a) is satisfied, and thus F is smooth.

Exercise 2 (*Smoothness is a local property*)**:** Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Prove the following assertions:

(a) If every point $p \in M$ has a neighborhood U such that $F|_U$ is smooth, then F is smooth.

(b) If F is smooth, then its restriction to every open subset of M is smooth.

Solution: Recall that (see *Example 1.10*(4)) any open subset U of M is considered as an open submanifold of M, endowed with the smooth structure $\overline{\mathcal{A}_U}$ determined by the smooth atlas

$$\mathcal{A}_U \coloneqq \big\{(W,\theta) \mid (W,\theta) \text{ is a smooth chart for M such that } W \subseteq U\big\}.$$

(a) Let $p \in M$. By hypothesis there exists an open neighborhood U of p in M such that $F|_U$ is smooth. By definition of smoothness, there are smooth charts $(W,\theta) \in \overline{\mathcal{A}_U}$ containing p and (V,ψ) containing $F(p)$ such that $F|_U(W) \subseteq V$ and $\psi \circ (F|_U) \circ \theta^{-1}$ is smooth from $\theta(W)$ to $\psi(V)$. But then (W,θ) is also a smooth chart for M containing p (with $W \subseteq U$) and $F(W) = F|_U(W) \subseteq V$. Since we also have

$$\psi \circ F \circ \theta^{-1} = \psi \circ (F|_U) \circ \theta^{-1}$$

on $\theta(W)$, we conclude that the former is smooth. As $p \in M$ was arbitrary, we infer that F is smooth.

(b) Let U be an open subset of M and let $p \in U$. By smoothness of F there exist smooth charts (W,θ) for M containing p and (V,ψ) for N containing $F(p)$ such that $F(W) \subseteq V$ and such that $\psi \circ F \circ \theta^{-1}$ is smooth from $\theta(W)$ to $\psi(V)$. Now, set $W' \coloneqq W \cap U$ and $\theta' \coloneqq \theta|_{W \cap U}$. Then (W',θ') is a smooth chart for U containing p, and we also have $F|_U(W') \subseteq F(W) \subseteq V$ and

$$\psi \circ (F|_U) \circ (\theta')^{-1} = (\psi \circ F \circ \theta^{-1})|_{\theta'(W')}.$$

Hence, $\psi \circ (F|_U) \circ (\theta')^{-1}$ is smooth from $\theta'(W')$ to $\psi(V)$. As $p \in U$ was arbitrary, we conclude that $F|_U$ is smooth.

Exercise 3: Let M, N and P be smooth manifolds. Prove the following assertions:

(a) If $c\colon M \to N$ is a constant map, then c is smooth.

(b) The identity map $\mathrm{Id}_M\colon M \to M$ is smooth.

(c) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota\colon U \hookrightarrow M$ is smooth.

(d) If $F\colon M \to N$ and $G\colon N \to P$ are smooth maps, then the composite $G \circ F\colon M \to P$ is also smooth.

Solution:

(a) Since c is constant, there exists a point $q \in N$ such that $c(x) = q$ for all $x \in M$. Fix $p \in M$, pick smooth charts (U, φ) containing p and (V, ψ) containing $q = c(p)$, and observe that $\{q\} = c(U) \subseteq V$. Since the composite map $\psi \circ c \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is clearly a constant map (with value $\psi(q)$) between open subsets of Euclidean spaces, it is certainly smooth. Therefore, the given constant map c is smooth.

(b) The identity map $\mathrm{Id}_M\colon M \to M$ of M clearly has an identity map between open subsets of Euclidean spaces as a coordinate representation, so it is smooth.

(c) Fix $p \in U \subseteq M$. Recall that a smooth chart for U containing p is simply a smooth chart (V, ψ) for M such that $p \in V \subseteq U$, and clearly it holds that $\iota(V) = V$. Since the coordinate representation of ι with respect to such a smooth chart is the identity map $\mathrm{Id}_{\psi(V)}\colon \psi(V) \to \psi(V)$, we deduce that $\iota\colon U \hookrightarrow M$ is smooth.

(d) Fix $p \in M$. Since G is smooth, there exist smooth charts (V, ψ) containing $F(p)$ and (W, θ) containing $G(F(p)) = (G \circ F)(p)$ such that $G(V) \subseteq W$ and the composite map

$$\theta \circ G \circ \psi^{-1}\colon \psi(V) \to \theta(W)$$

is smooth. Since F is smooth, it is continuous by *Proposition 2.5*, so $F^{-1}(V)$ is an open neighborhood of p in M, and thus there exists a smooth chart (U, φ) for M such that $p \in U \subseteq F^{-1}(V)$. In addition, the composite map

$$\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$$

is smooth by *Remark 2.7*, and we also have $(G \circ F)(U) \subseteq G(V) \subseteq W$. Now, observe that

$$\theta \circ (G \circ F) \circ \varphi^{-1} = \left(\theta \circ G \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \varphi^{-1}\right)\colon \varphi(U) \to \theta(W)$$

is smooth as a composition of smooth maps between open subsets of Euclidean spaces. Hence, the composite map $G \circ F\colon M \to P$ is smooth, as claimed.

Exercise 4: Let $M_1, \ldots, M_k$ be smooth manifolds. For each $i \in \{1, \ldots, k\}$, let

$$\pi_i\colon \prod_{j=1}^{k} M_j \to M_i$$

be the projection onto the i-th factor.

(a) Show that each π_i is smooth.

(b) Let N be a smooth manifold. Show that a map $F\colon N \to \prod_{j=1}^k M_j$ is smooth if and only if each of the component maps $F_i := \pi_i \circ F\colon N \to M_i$ is smooth.

Solution:

(a) Let $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k =: M$ and $i \in \{1, \ldots, k\}$ be arbitrary. Let (U_i, φ_i) be a smooth chart containing i. By the construction in [*Exercise Sheet* 2, *Exercise* 3] the smooth structure of M is generated by products of smooth charts of the individual factors. Hence, if for $j \neq i$ we take some smooth chart (U_j, φ_j) for M_j containing p_j and write $U = U_1 \times \ldots \times U_k$, resp. $\varphi = \varphi_1 \times \ldots \times \varphi_k$, then we obtain that (U, φ) is a smooth chart for M containing p. Note then that $\pi_i(U) \subseteq U_i$, and thus the coordinate representation $\widehat{\pi}_i = \varphi_i \circ \pi_i \circ \varphi^{-1}$ of π_i is a map from $\varphi_1(U_1) \times \ldots \times \varphi_k(U_k)$ to $\varphi_i(U_i)$. Furthermore, it is straightforward to see that for all $(v_1, \ldots, v_k) \in \varphi_1(U_1) \times \ldots \times \varphi_k(U_k) \subseteq \mathbb{R}^n$ (where $n := n_1 + \ldots + n_k$), we have

$$\widehat{\pi}_i(v_i) = \varphi_i \circ \pi_i \circ \varphi^{-1}(v_1, \ldots, v_k) = v_i,$$

and thus $\widehat{\pi}_i$ is the projection to the i-th factor $\varphi_1(U_1) \times \cdots \times \varphi_k(U_k) \to \varphi_i(U_i)$. In particular, it is smooth. As $p \in M$ was arbitrary, we conclude that the definition of smoothness is satisfied by π_i; in other words, π_i it is smooth, as claimed.

(b) Suppose first that $F\colon N \to \prod_{j=1}^k M_j$ is smooth. Pick $1 \leq i \leq k$. By (a) we know that π_i is smooth, and by *Exercise* 3(d) we know that a composition of smooth maps is smooth. Hence, $F_i = \pi_i \circ F$ is smooth.

Suppose now that each of the component maps $F_i = \pi_i \circ F$ is smooth. Let $q \in N$ and set $F(q) = (p_1, \ldots, p_k)$, so that $p_i = F_i(q)$. By hypothesis, for every $1 \leq i \leq k$ there exist smooth charts (V_i, ψ_i) for N containing q and (U_i, φ_i) for M_i containing p_i such that $F_i(V_i) \subseteq U_i$ and such that $\varphi_i \circ F_i \circ \psi_i^{-1}$ is smooth from $\psi_i(V_i)$ to $\varphi_i(U_i)$. Set $V := V_1 \cap \ldots \cap V_k$ and observe that this is an open neighborhood of q. Now, fix any $1 \leq i \leq k$ and set $\psi = \psi_i|_V$. Note that $F_j(V) \subseteq U_j$ for all $1 \leq j \leq k$, so by *Remark 2.7* we infer that $\varphi_j \circ F_j \circ \psi^{-1}$ is smooth from $\psi(V)$ to $\varphi_j(U_j)$ for all j. Moreover, we have

$$F(V) \subseteq F_1(V_1) \times \ldots \times F_k(V_k) \subseteq U_1 \times \ldots \times U_k.$$

In summary, (V, ψ) is a smooth chart for N containing q and $(U_1 \times \ldots \times U_k, \varphi_1 \times \ldots \times \varphi_k)$ is a smooth chart for $M_1 \times \ldots \times M_k$ containing $F(q)$ such that $F(V) \subseteq U_1 \times \ldots \times U_k$, and the coordinate representation

$$(\varphi_1 \times \ldots \times \varphi_k) \circ F \circ \psi^{-1} = (\varphi_1 \circ F_1 \circ \psi^{-1}) \times \ldots \times (\varphi_k \circ F_k \circ \psi^{-1})$$

is smooth from $\psi(V)$ to $\varphi_1(U_1) \times \ldots \times \varphi_k(U_k)$, because all of its components are smooth. As $q \in N$ was arbitrary, we conclude that F is smooth.

Exercise 5 (to be submitted):

(a) Consider the canonical inclusion $\iota\colon \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$ and the graph coordinates

$$\left\{(U_i^{\pm} \cap \mathbb{S}^1, \varphi_i^{\pm})\right\}_{i=1}^2$$

for unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. Compute all possible coordinate representations of ι and deduce that the differential $d\iota_w$ is injective for any $w \in \mathbb{S}^1$.

(b) Show that $\mathbb{RP}^1 \cong \mathbb{S}^1$ as smooth manifolds.

[Hint: To define an appropriate map from $\mathbb{RP}^1$ to $\mathbb{S}^1$, it might be helpful to use the identifications $\mathbb{R}^2 \cong \mathbb{C}$ and $\mathbb{S}^1 \cong \{z \in \mathbb{C} \mid |z| = 1\}$, and to check its smoothness, *Exercise A*.6 might be useful.]

Solution:

(a) Denote by e_1, e_2 the standard basis of $\mathbb{R}^2$. It is then straightforward to verify that the coordinate representation $\iota \circ (\varphi_i^{\pm})^{-1}$ of ι is given by

$$\begin{aligned} \iota \circ (\varphi_i^{\pm})^{-1} \colon (-1,1) &\to \mathbb{R}^2 \\ t &\mapsto t\, e_{3-i} \pm \sqrt{1-t^2}\, e_i. \end{aligned}$$

The differential at a given $t \in (-1,1)$ is hence given by

$$\begin{aligned} d\big(\iota \circ (\varphi_i^{\pm})^{-1}\big)_t \colon \mathbb{R} &\to \mathbb{R}^2 \\ x &\mapsto x \left(e_{3-i} \mp \frac{2t}{\sqrt{1-t^2}} e_i \right), \end{aligned}$$

which is clearly injective. By the functoriality of the differential, we have

$$d\big(\iota \circ (\varphi_i^{\pm})^{-1}\big)_t = d\iota_{(\varphi_i^{\pm})^{-1}(t)} \circ d\big((\varphi_i^{\pm})^{-1}\big)_t,$$

and $d\big((\varphi_i^{\pm})^{-1}\big)_t$ is an invertible linear map (again by functoriality). Thus, $d\iota_{(\varphi_i^{\pm})^{-1}(t)}$ is injective for any $t \in (-1,1)$ and any choice of chart, so $d\iota_w$ is injective for any $w \in \mathbb{S}^1$.

(b) We construct a map from $\mathbb{RP}^1$ to $\mathbb{S}^1$ which at the end should be the desired diffeomorphism. To this end, we first identify $\mathbb{C}$ with $\mathbb{R}^2$, and thus $\mathbb{R}^2 \setminus \{0\} = \mathbb{C}^\times$, and $\mathbb{S}^1$ with $\{z \in \mathbb{C} \mid |z| = 1\}$. Note next that the map

$$\begin{aligned} F \colon \mathbb{C}^\times &\to \mathbb{S}^1 \\ z &\mapsto \left(\frac{z}{|z|}\right)^2 \end{aligned}$$

is smooth and invariant under scaling by a non-zero real number. Therefore, we have an induced map

$$f \colon \mathbb{RP}^1 \to \mathbb{S}^1$$

such that $f \circ \pi = F$, where $\pi \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1$ is the quotient map. Hence, f is smooth by *Exercise A*.6.

To prove that f is a diffeomorphism, we manually construct an inverse and show that it is smooth. Since for f we take the square of a complex number, we want to take the square root to construct its inverse. We thus switch to exponential notation, and define

$$\begin{aligned} g \colon \mathbb{S}^1 &\to \mathbb{RP}^1 \\ e^{it} &\mapsto [e^{it/2}]. \end{aligned}$$

It is straightforward to check that g is well-defined set-theoretically, because adding $2\pi n$ to t only changes $e^{it}/2$ up to a sign.

Let us now show that g is smooth by computing the appropriate coordinate representations. We have

$$\begin{aligned}\varphi_1 \circ g \circ \varphi_1^{\pm} \colon (-1,1) &\to \mathbb{R}\\ t &\mapsto \frac{\sin(\pm \arcsin(t)/2)}{\cos(\pm \arcsin(t)/2)}\end{aligned}$$

and

$$\begin{aligned}\varphi_2 \circ g \circ \varphi_2^{\pm} \colon (-1,1) &\to \mathbb{R}\\ t &\mapsto \frac{\cos(\pm \arccos(t)/2)}{\sin(\pm \arccos(t)/2)}.\end{aligned}$$

These are smooth since they are fractions of compositions of smooth functions, with non-zero denominator. Hence, g is smooth.

It remains to check that f and g are mutually inverse. To this end, we compute that

$$(f \circ g)(e^{it}) = f\big([e^{it/2}]\big) = F(e^{it/2}) = (e^{it/2})^2 = e^{it}$$

and

$$(g \circ f)([e^{it}]) = g(e^{2it}) = [e^{it}].$$

This concludes the solution.

Remark: Later in the course, the more principal approach is to show that f is smooth, bijective and a local diffeomorphism, which – as we will later see – is equivalent to being a diffeomorphism. Like this, we avoid having to construct an inverse by hand. This is how we would do it:

Let us start by proving that f is bijective.

- Surjectivity: Let $w \in \mathbb{S}^1$ be arbitrary. Then there exists $z \in \mathbb{C}$ with $z^2 = w$. In particular, we have $|z| = 1$, and thus $f([z]) = F(z) = w$.
- Injectivity: Suppose that $f([z]) = f([z'])$ for some $[z], [z'] \in \mathbb{RP}^1$. Then

$$(z/|z|)^2 = (z'/|z'|)^2 \implies z/|z| = \pm z'/|z'| \implies [z] = [z'].$$

To conclude, we show that it is a local diffeomorphism, which is equivalent to proving that $df_{[z]}$ is an isomorphism for all $[z] \in \mathbb{RP}^1$. As we work in dimension 1, this amounts to showing that $df_{[z]} \neq 0$ for all $[z] \in \mathbb{RP}^1$. Now, if $\iota \colon \mathbb{S}^1 \hookrightarrow \mathbb{C}$ denotes the canonical inclusion, then we have

$$G \coloneqq \iota \circ F = \iota \circ f \circ \pi \implies dG_z = d\iota_{f([z])} \circ df_{[z]} \circ d\pi_z$$

for any $z \in \mathbb{C}^\times$. As $d\iota_w$ is injective for any $w \in \mathbb{S}^1$ by part (a), it suffices to prove that $dG_z \neq 0$ to obtain that $df_{[z]} \neq 0$. Note that, under our identifications, $G = \iota \circ f$ is the function

$$\begin{aligned}G \colon \mathbb{R}^2 \setminus \{0\} &\to \mathbb{R}^2\\ (x,y) &\mapsto \Big((x+iy)/\sqrt{x^2+y^2}\Big)^2 = \left(\frac{x^2-y^2}{x^2+y^2}, \frac{2xy}{x^2+y^2}\right)\end{aligned}$$

whose Jacobian is thus given by

$$dG_{(x,y)} = \begin{pmatrix} \dfrac{4xy^2}{(x^2+y^2)^2} & \dfrac{-4xy^2}{(x^2+y^2)^2} \\ \dfrac{2y(y^2-x^2)}{x^2+y^2} & \dfrac{2x(x^2-y^2)}{x^2+y^2} \end{pmatrix}$$

This is clearly non-zero for any $(x, y) \neq (0, 0)$. Hence, f is a diffeomorphism; in other words, $\mathbb{RP}^1 \cong \mathbb{S}^1$.