Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 2 – Solutions

Exercise 1 (*Finite-dimensional vector spaces*)**:** Let V be an $\mathbb{R}$-vector space of dimension n. Recall that any norm on V determines a topology, which is independent of the choice of norm. Show that V has a natural smooth manifold structure as follows:

(a) Pick a basis $E_1, \dots, E_n$ for V and consider the map

$$E\colon \mathbb{R}^n \to V,\ (x^1, \dots, x^n) \mapsto \sum_{i=1}^{n} x^i E_i.$$

Show that (V, E^{-1}) is a chart for V; in particular, with the topology defined above, V is thus a topological n-manifold.

(b) Given a different basis $\widetilde{E}_1, \dots, \widetilde{E}_n$ for V, show that the charts (V, E^{-1}) and $(V, \widetilde{E}^{-1})$ are smoothly compatible.

The collection of all such charts for V defines a smooth structure, called the *standard smooth structure on V*.

Solution: Denote by $\|\bullet\|_V : V \to \mathbb{R}_{\geq 0}$ the chosen norm on V, and by $\|\bullet\|_{\mathbb{R}^n}$ the standard Euclidean norm on $\mathbb{R}^n$.

(a) It suffices to show that E is a homeomorphism. First, observe that E is bijective. Now, note that the map $\|\cdot\|'\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by $\|x\|' \coloneqq \|E(x)\|_V$ is a norm on $\mathbb{R}^n$. As all norms on $\mathbb{R}^n$ are equivalent, there exists a constant $c > 1$ such that

$$\frac{1}{c}\|x\|_{\mathbb{R}^n} \leq \|x\|' \leq c\|x\|_{\mathbb{R}^n} \quad \text{for all } x \in \mathbb{R}^n.$$

In particular, both E and E^{-1} are Lipschitz-continuous, and thus E is a homeomorphism.

(b) There exists an invertible matrix $A = (A_i^j)_{1\leq i,j\leq n} \in \mathrm{GL}(n, \mathbb{R})$ such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. Thus, the transition map $\widetilde{E}^{-1} \circ E$ is given by

$$\big(\widetilde{E}^{-1} \circ E\big)(x) = \widetilde{E}^{-1}\Big(\sum_i x^i E_i\Big) = \sum_i x^i \widetilde{E}^{-1}(E_i) = \sum_i x^i A_i^j.$$

Hence, the transition map is an invertible linear map, and thus a diffeomorphism (the partial derivatives of the first order are given by constant maps corresponding to the entries of A, and the partial derivatives of higher order vanish).

Exercise 2: Prove the following assertions:

(a) The space $M(m \times n, \mathbb{R})$ of $m \times n$ matrices with real entries has a natural smooth manifold structure.

(b) The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ (i.e., the group of invertible $n \times n$ matrices with real entries) has a natural smooth manifold structure.

(c) The subset $M_m(m \times n, \mathbb{R})$ of $M(m \times n, \mathbb{R})$ of matrices of rank m, where $m < n$ has a natural smooth manifold structure. Similarly for $M_n(m \times n, \mathbb{R})$ when $n < m$.

(d) The space $\mathcal{L}(V, W)$ of $\mathbb{R}$-linear maps from V to W, where V and W are two finite-dimensional $\mathbb{R}$-vector spaces, has a natural smooth manifold structure.

What is the dimension of each of the above smooth manifolds?

Solution:

(a) The set $M(m \times n, \mathbb{R})$ is an $\mathbb{R}$-vector space of dimension mn, and thus by *Exercise* 1 it has a natural smooth structure, given by identifying it with $\mathbb{R}^{mn}$. We have

$$\dim M(m \times n, \mathbb{R}) = mn.$$

(b) Let $\det\colon M(n \times n, \mathbb{R}) \to \mathbb{R}$ be the determinant function. Note that it is continuous, and hence $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is an open subset of $M(n \times n, \mathbb{R})$. As the latter has a natural smooth manifold structure by (a), the open subset $\mathrm{GL}(n, \mathbb{R}) \subseteq M(n \times n, \mathbb{R})$ inherits a natural smooth manifold structure as well. We have

$$\dim \mathrm{GL}(n, \mathbb{R}) = n^2.$$

(c) By linear algebra we know that an $m \times n$-matrix with $m < n$ has full rank if and only if it has an invertible $m \times m$-submatrix. For a subset $I \subseteq \{1, \ldots, n\}$ of cardinality m and a matrix $A \in M(m \times n, \mathbb{R})$, denote by A_I the $m \times m$ submatrix corresponding to the columns indexed by I. Consider the map

$$\begin{aligned} \det_I\colon M(m \times n, \mathbb{R}) &\to \mathbb{R} \\ A &\mapsto \det(A_I) \end{aligned}$$

and observe that it is continuous. Thus,

$$M_m(m \times n, \mathbb{R}) = \bigcup_{\substack{I \subseteq \{1, \ldots, n\} \\ |I| = m}} \det_I^{-1}(\mathbb{R} \setminus \{0\})$$

is an open subset of $M(m \times n, \mathbb{R})$, and hence $M_m(m \times n, \mathbb{R})$ inherits a natural smooth manifold structure. We have

$$\dim M_m(m \times n, \mathbb{R}) = mn.$$

Finally, note that the isomorphism of vector spaces $M(m \times n, \mathbb{R}) \to M(n \times m, \mathbb{R})$ given by transposition preserves the rank. Therefore, $M_n(m \times n, \mathbb{R})$ with $n < m$ is again open in $M(m \times n, \mathbb{R})$, and hence inherits a natural smooth manifold structure.

(d) The set $\mathcal{L}(V, W)$ is naturally an $\mathbb{R}$-vector space, and hence it has a natural smooth manifold structure by *Exercise* 1. Indeed, fixing bases of V and W, $\mathcal{L}(V, W)$ can be naturally identified with $M(m \times n, \mathbb{R})$, where $m = \dim W$ and $n = \dim V$. We have

$$\dim \mathcal{L}(V, W) = \dim V \cdot \dim W.$$

Exercise 3 (*Product manifolds*)**:** Let $M_1, \ldots, M_k$ be smooth manifolds of dimensions $n_1, \ldots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \ldots \times M_k$ is a smooth manifold of dimension $n_1 + \ldots + n_k$ by constructing a smooth manifold structure on it.

Solution: By [*Exercise Sheet* 1, *Exercise* 5] we know that $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$. As in the solution of [*Exercise Sheet* 1, *Exercise* 5], we see that if $\mathcal{A}_i$ denotes the smooth structure of M_i for $1 \leq i \leq k$, then

$$\mathcal{A} \coloneqq \left\{ (U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k) \mid (U_1, \varphi_1) \in \mathcal{A}_1, \ldots, (U_k, \varphi_k) \in \mathcal{A}_k \right\}$$

is an atlas for $M_1 \times \ldots \times M_k$. To see that it is smooth, observe that the transition map between two charts $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$ and $(V_1 \times \cdots \times V_k, \psi_1 \times \cdots \times \psi_k)$ of $\mathcal{A}$ is given by

$$(\psi_1 \circ \varphi_1^{-1}) \times \ldots \times (\psi_k \circ \varphi_k^{-1}),$$

which is smooth, since each factor is smooth. Therefore, $\mathcal{A}$ is a smooth atlas, and hence determines a smooth structure on $M_1 \times \ldots \times M_k$ by *Proposition 1.8*(a).

Exercise 4 (*The real projective n-space*)**:** Denote by $\mathbb{RP}^n$ the real projective n-space, where $n \geq 1$. For each $i \in \{0, \ldots, n\}$ consider the chart (U_i, φ_i) for $\mathbb{RP}^n$ described in Appendix A, where

$$U_i \coloneqq \left\{ [x_0 : \ldots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0 \right\} \subseteq \mathbb{RP}^n$$

and

$$\begin{aligned} \varphi_i \colon U_i &\to \mathbb{R}^n \\ [x_0 : \ldots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right). \end{aligned}$$

Show that the collection

$$\mathcal{A}_{\mathbb{RP}^n} \coloneqq \left\{ (U_i, \varphi_i) \right\}_{i=0}^n$$

is a smooth atlas for $\mathbb{RP}^n$.

Solution: Let $0 \leq i < j \leq n$. One can check that

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} \colon \mathbb{R}^n_{x_j \neq 0} &\to \mathbb{R}^n_{x_{i+1} \neq 0} \\ (x_1, \ldots, x_n) &\mapsto \frac{1}{x_j} \left(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \right), \end{aligned}$$

and

$$\begin{aligned}\varphi_i \circ \varphi_j^{-1} \colon \mathbb{R}^n_{x_{i+1}\neq 0} &\to \mathbb{R}^n_{x_j \neq 0}\\ (x_1, \dots, x_n) &\mapsto \frac{1}{x_{i+1}} \left(x_1, \dots, x_i, x_{i+2}, \dots, x_j, 1, x_{j+1}, \dots, x_n\right),\end{aligned}$$

which demonstrates that the transition functions are smooth. This proves the assertion.

Exercise 5 (to be submitted): Consider the n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. Denote by $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ the *north pole* and by $S = -N = (0, \dots, 0, -1)$ the *south pole* of $\mathbb{S}^n$. Define the *stereographic projection from the north pole N* as follows:

$$\sigma \colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \quad \sigma(x^1, \dots, x^{n+1}) = \frac{1}{1 - x^{n+1}} \, (x^1, \dots, x^n).$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$; it is called the *stereographic projection from the south pole.*

(a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.

(b) Show that σ is bijective, and

$$\sigma^{-1}(u^1, \dots, u^n) = \frac{1}{|u|^2 + 1} \, (2u^1, \dots, 2u^n, |u|^2 - 1).$$

(c) Verify that the atlas consisting of the two charts $\big(\mathbb{S}^n \setminus \{N\}, \sigma\big)$ and $\big(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma}\big)$ is a smooth atlas for $\mathbb{S}^n$, and hence defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called *stereographic coordinates.*)

(d) Show that the smooth structure determined by the above atlas is the same as the one defined via graph coordinates in *Example 1.10*(2).

Solution: Denote by H the linear subspace of $\mathbb{R}^{n+1}$ where $x^{n+1} = 0$, i.e.,

$$H = \big\{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^{n+1} = 0\big\},$$

and observe that H can be identified with $\mathbb{R}^n$.

(a) The line $\ell_1 \subseteq \mathbb{R}^{n+1}$ through $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ and $x = (x^1, \dots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$ is given by the parametric equation

$$\ell_1 : (x^1, \dots, x^{n+1}) + t(-x^1, \dots, -x^n, 1 - x^{n+1}), \; t \in \mathbb{R}$$

and intersects the hyperplane $H : (x^{n+1} = 0)$ for $t = -\frac{x^{n+1}}{1 - x^{n+1}}$ at the point $(u, 0) \in \mathbb{R}^{n+1}$, where

$$\begin{aligned} u &= \left(x^1 + \frac{x^{n+1}}{1 - x^{n+1}} x^1, \dots, x^n + \frac{x^{n+1}}{1 - x^{n+1}} x^n\right) \\ &= \left(\frac{x^1}{1 - x^{n+1}}, \dots, \frac{x^n}{1 - x^{n+1}}\right) = \sigma(x) \in \mathbb{R}^n. \end{aligned}$$

Similarly, we see that

$$\widetilde{\sigma}(x) = -\sigma(-x) = -\Big(-\frac{x^1}{1+x^{n+1}}, \dots, -\frac{x^n}{1+x^{n+1}}\Big) = \Big(\frac{x^1}{1+x^{n+1}}, \dots, \frac{x^n}{1+x^{n+1}}\Big)$$

is the point where the line $\ell_2 \subseteq \mathbb{R}^{n+1}$ through S and x intersects the hyperplane H.

(b) Pick a point $u = (u^1, \dots, u^n) \in \mathbb{R}^n$. The line $\ell \subseteq \mathbb{R}^{n+1}$ through $(u, 0) \in \mathbb{R}^{n+1}$ and $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ is given by the parametric equation

$$\ell : (u^1, \dots, u^n, 0) + t(-u^1, \dots, -u^n, 1), \ t \in \mathbb{R}$$

and intersects the n-sphere $\mathbb{S}^n$ at points which satisfy the equation

$$|u|^2(1-t)^2 + t^2 = 1,$$

where $|u|^2 = \sum_{i=1}^n (u^i)^2$. It is now easy to check that the above equation has two solutions: $t = 1$, which corresponds to the point $N \in \mathbb{S}^n$, and $t = \frac{|u|^2-1}{|u|^2+1} \neq 1$, which corresponds to the point

$$x = \Big(\frac{2u^1}{|u|^2+1}, \dots, \frac{2u^n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\Big) \in \mathbb{S}^n.$$

Therefore, the map

$$\sigma \colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \quad (x^1, \dots, x^{n+1}) \mapsto \frac{1}{1-x^{n+1}}\, (x^1, \dots, x^n)$$

is bijective, and its inverse $\sigma^{-1} \colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$ is given by the formula

$$(u^1, \dots, u^n) \mapsto \frac{1}{|u|^2+1}\, (2u^1, \dots, 2u^n, |u|^2 - 1).$$

(c) It is straightforward to check that

$$\big(\widetilde{\sigma} \circ \sigma^{-1}\big)(u^1, \dots, u^n) = \frac{1}{|u|^2}(u^1, \dots, u^n), \ (u^1, \dots, u^n) \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}$$

and that its inverse $\sigma \circ \widetilde{\sigma}^{-1}$ is also given by the same formula, namely,

$$\big(\sigma \circ \widetilde{\sigma}^{-1}\big)(u^1, \dots, u^n) = \frac{1}{|u|^2}(u^1, \dots, u^n), \ (u^1, \dots, u^n) \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}.$$

Since both $\widetilde{\sigma} \circ \sigma^{-1}$ and $\sigma \circ \widetilde{\sigma}^{-1}$ are clearly smooth, the two charts $\big(\mathbb{S}^n \setminus \{N\}, \sigma\big)$ and $\big(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma}\big)$ for $\mathbb{S}^n$ are smoothly compatible, and since their domains clearly cover $\mathbb{S}^n$, they comprise a smooth atlas for $\mathbb{S}^n$, which determines a smooth structure on $\mathbb{S}^n$ by *Proposition 1.8* (a).

(d) According to *Proposition 1.8* (b), to prove the claim, we have to check that the graph coordinates $\big(U_i^\pm \cap \mathbb{S}^n, \varphi_i^\pm\big)$, where

$$\varphi_i^\pm \colon U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n, \ (x^1, \dots, x^{n+1}) \mapsto (x^1, \dots, \widehat{x^i}, \dots, x^{n+1})$$

with inverse

$$\left(\varphi_i^{\pm}\right)^{-1} \colon \mathbb{B}^n \to U_i^{\pm} \cap \mathbb{S}^n,\ (u^1, \dots, u^n) \mapsto \left(u^1, \dots, \pm\sqrt{1-|u|^2}, \dots, u^n\right),$$

and the stereographic coordinates $\left(\mathbb{S}^n \backslash \{N\}, \sigma\right)$ and $\left(\mathbb{S}^n \backslash \{S\}, \widetilde{\sigma}\right)$ are smoothly compatible. For instance, we have

$$\left(\varphi_i^{\pm} \circ \sigma^{-1}\right)(u^1, \dots, u^n) = \left(\frac{2u^1}{1+|u|^2}, \dots, \widehat{\frac{2u^i}{1+|u|^2}}, \dots, \frac{2u^n}{1+|u|^2}, \frac{|u|^2-1}{1+|u|^2}\right)$$

for $1 \le i \le n$, and

$$\left(\varphi_{n+1}^{\pm} \circ \sigma^{-1}\right)(u^1, \dots, u^n) = \left(\frac{2u^1}{1+|u|^2}, \dots, \frac{2u^i}{1+|u|^2}, \dots, \frac{2u^n}{1+|u|^2}\right),$$

which are clearly smooth in their domain of definition. In a similar fashion one can readily verify that the remaining charts are smoothly compatible; this yields the assertion.