



Differential Geometry II - Smooth Manifolds

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Lecturer: Dimitri Stelio Wyss

Assistant: Eric Yen-Yo Chen

Exercise Sheet 1

Exercise 1:

Examine which of the following spaces (endowed with the subspace topology) is locally Euclidean:

- (a) The closed interval $[0, 1] \subseteq \mathbb{R}$.
- (b) The “bent line” $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, xy = 0\} \subseteq \mathbb{R}^2$.

Solution:

(a) The interval $[0, 1]$ is *not* locally Euclidean. Suppose by contradiction that it is locally Euclidean. By *Exercise 1* there is a neighborhood $U \subseteq [0, 1]$ of 0 which is homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$. Denote by $\varphi: U \rightarrow \mathbb{R}^n$ a homeomorphism and note that U is connected, and thus of the form $U = [0, \varepsilon)$ for some $\varepsilon > 0$. But then $U \setminus \{0\} = (0, \varepsilon)$ is homeomorphic to $\mathbb{R}^n \setminus \{\varphi(0)\}$, and since $(0, \varepsilon)$ is still connected, we infer that $n > 1$ ($\mathbb{R}$ minus a point has two connected components). Now there are two ways to conclude: First, note that $(0, \varepsilon)$ and $\mathbb{R}^n \setminus \{\varphi(0)\}$ are topological manifolds of dimension 1 and n , respectively, and since the dimension of a topological manifold is a topological invariant, we obtain $n = 1$, a contradiction. Second, if $x \in (0, \varepsilon)$, then $(0, \varepsilon) \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n \setminus \{\varphi(0), \varphi(x)\}$; as $n > 1$, the latter is connected, while the former is not, a contradiction.

(b) The “bent line”

$$L := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, xy = 0\}$$

is locally Euclidean. Indeed, denote by $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the counterclockwise rotation around the origin by 45° . As this is a homeomorphism, we obtain that $L \cong \varphi(L)$. But now note that $\varphi(L)$ coincides with the graph of the absolute value function $|\bullet|: \mathbb{R} \rightarrow \mathbb{R}$. Thus, we obtain $L \cong \varphi(L) \cong \mathbb{R}$.

Exercise 2:

(a) *The line with two origins*: Consider the set

$$X = \{(x, y) \in \mathbb{R}^2 \mid y \in \{-1, 1\}\} \subseteq \mathbb{R}^2$$

and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff.

(b) Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Solution:

(a) Denote by $\pi: X \rightarrow M$ the quotient map $(x, y) \mapsto [(x, y)]$. The two “origins” are the equivalence classes of the points $(0, y) \in X$ for $y = \pm 1$; these classes have just one element each and we denote them by $0_y = [(0, y)] = \{(0, y)\} \in M$. In contrast, the equivalence class of any other point $(x, y) \in X$ with $x \neq 0$ is the two-point set $\tilde{x} = [(x, y)] = \{(x, 1), (x, -1)\} \in M$. Therefore, M is the set of equivalence classes

$$M = X / \sim = \{0_1\} \cup \{0_{-1}\} \cup \{\tilde{x}\}_{x \neq 0}.$$

The space M is locally Euclidean of dimension 1 because it is the union of two open sets

$$\mathbb{R}_y = \{[(x, y)] \in M \mid x \in \mathbb{R}\} \quad (\text{for } y = \pm 1),$$

each of which is homeomorphic to $\mathbb{R}$ via the map

$$\begin{aligned} \varphi_y: \mathbb{R} &\rightarrow \mathbb{R}_y \\ x &\mapsto [(x, y)]. \end{aligned}$$

To see that the sets $\mathbb{R}_y$ are open in the quotient topology, note that

$$\pi^{-1}(\mathbb{R}_y) = X \setminus \{(0, -y)\},$$

which is open in X .

Moreover, M is second-countable because it is the union of two second-countable open subsets, namely, the sets $\mathbb{R}_y \cong \mathbb{R}$ (for $y = \pm 1$).

Finally, M is not Hausdorff: let U_{-1} be any open set containing 0_{-1} and let U_1 be any open set containing 0_1 . For $y \in \{-1, 1\}$, as $\pi^{-1}(U_y)$ is an open subset of X containing $(0, y)$, it contains a set of the form $V_y = (-\varepsilon_y, \varepsilon_y) \times \{y\}$ for some $\varepsilon_y > 0$. Now let x be a real number such that $0 < x < \min\{\varepsilon_{-1}, \varepsilon_1\}$. Then $[(x, -1)] = [(x, 1)]$ is contained in both U_{-1} and U_1 . Hence, 0_{-1} and 0_1 cannot be separated by disjoint open neighborhoods.

(b) Let I be an uncountable index set. For every $i \in I$ denote by $\mathbb{R}_i$ a copy of the real numbers $\mathbb{R}$ equipped with the Euclidean topology, and let

$$X := \bigsqcup_{i \in I} \mathbb{R}_i$$

be their disjoint union. Recall that there is a natural topology on X , defined as follows: For every i , denote by $f_i: \mathbb{R}_i \rightarrow X$ the natural set-theoretic inclusion. Then

$$\tau := \{U \subseteq X \mid \forall i \in I : f_i^{-1}(U) \text{ open in } \mathbb{R}_i\}$$

is a topology on X ; in fact, it is the finest (i.e., maximal) topology on X such that all the maps f_i are continuous.

To see that (X, τ) is Hausdorff, let $x, y \in X$ be arbitrary. Let $i, j \in I$ be such that $x \in f_i(\mathbb{R}_i)$ and $y \in f_j(\mathbb{R}_j)$. If $i \neq j$, then $f_i(\mathbb{R}_i)$ and $f_j(\mathbb{R}_j)$ are disjoint open neighborhoods of x and y , respectively (check this!). If $i = j$, then since $\mathbb{R}_i$ is Hausdorff, we can find disjoint open neighborhoods $U, V \subseteq \mathbb{R}_i$ separating (the preimages of) x and y in $\mathbb{R}_i$. Then $f_i(U)$ and $f_i(V)$ are disjoint open neighborhoods of x and y , respectively, inside X (again, check this!). As $x, y \in X$ were arbitrary, we conclude that X is Hausdorff.

Next, to check that X is locally Euclidean, let $x \in X$ be arbitrary. Let $i \in I$ be such that $x \in f_i(\mathbb{R}_i)$. Then $f_i(\mathbb{R}_i) \cong \mathbb{R}$ is a Euclidean open neighborhood of x inside X .

Finally, suppose by contradiction that X is second-countable, i.e. there exists a countable basis $\mathfrak{B}$ for its topology τ . Note that, for every $i \in I$, the set $f_i(\mathbb{R}_i)$ is open in X , and thus there exists $\emptyset \neq U_i \in \mathfrak{B}$ such that $U_i \subseteq f_i(\mathbb{R}_i)$. But then we must have $U_i \neq U_j$ for all $i \neq j$, and thus the map

$$I \rightarrow \mathfrak{B}, i \mapsto U_i$$

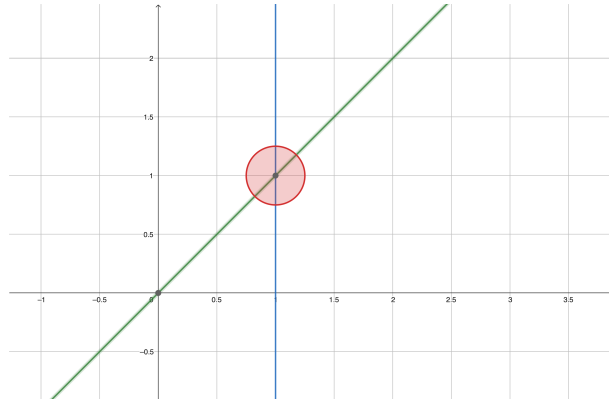
is an injection. However, since I is uncountable, this contradicts our hypothesis that $\mathfrak{B}$ is countable.

Exercise 3: Consider the subset

$$V = \{(x, y) \in \mathbb{R}^2 \mid (x - 1)(x - y) = 0\} \subseteq \mathbb{R}^2$$

endowed with the subspace topology. Show that V is not a topological manifold.

Solution: The subset $V \subseteq \mathbb{R}^2$ and a disc with small radius and centered at the point $(1, 1) \in \mathbb{R}^2$ (which is the point of intersection of the lines $y = x$ and $x = 1$) have been plotted below.



Since V is a subspace of $\mathbb{R}^2$, it is Hausdorff and second-countable. By considering any point $p \in V \setminus \{(1, 1)\}$, we conclude that if V were a topological manifold, then it would necessarily have dimension 1. Assume now by contradiction that V is a topological 1-manifold. Then there exists an open neighborhood W of $(1, 1)$ which is homeomorphic to an open subset G of $\mathbb{R}$; denote by φ this homeomorphism. For sufficiently small $\varepsilon > 0$, the set $U := B((1, 1), \varepsilon) \cap W$ (the red disc above) is an open neighborhood of $(1, 1)$ in W , which is connected. Hence, its homeomorphic image $I := \varphi(U)$ in $G \subseteq \mathbb{R}$ is connected as

well, and thus $I \subseteq \mathbb{R}$ is an open interval. Observe now that $U \setminus \{(1, 1)\}$ has four connected components, whereas $I \setminus \{\varphi(1, 1)\}$ has only two connected components, a contradiction. In conclusion, V is not a topological manifold.

Exercise 4 (*Product manifolds*):

Let $M_1, \dots, M_k$ be topological manifolds of dimensions $n_1, \dots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \dots \times M_k$ is a topological manifold of dimension $n_1 + \dots + n_k$.

Solution: Any finite product of Hausdorff spaces is also Hausdorff: two distinct points of the product differ at some coordinate, where we can separate them by two disjoint open neighborhoods. Furthermore, if for each $1 \leq i \leq k$ we denote by $\mathcal{B}_i$ a countable basis for the topology of M_i , then

$$\mathcal{B} := \{B_1 \times \dots \times B_k \mid \forall 1 \leq i \leq k : B_i \in \mathcal{B}_i\}$$

is a countable basis for the topology of the product $M_1 \times \dots \times M_k$. Finally, given any point $P = (p_1, \dots, p_k) \in M_1 \times \dots \times M_k$, by *Exercise 1* we know that for every $1 \leq i \leq k$ there exists an open neighborhood $U_i \subseteq M_i$ of p_i such that $U_i \cong \mathbb{R}^{n_i}$. Hence, $U := U_1 \times \dots \times U_k$ is an open neighborhood of P such that $U \cong \mathbb{R}^{n_1 + \dots + n_k}$. In conclusion, $M_1 \times \dots \times M_k$ is a topological manifold of dimension $n_1 + \dots + n_k$.

Exercise 5 (*Finite-dimensional vector spaces*):

Let V be an $\mathbb{R}$ -vector space of dimension n . Recall that any norm on V determines a topology, which is independent of the choice of norm. Show that V has a natural smooth manifold structure as follows:

- (a) Pick a basis $E_1, \dots, E_n$ for V and consider the map

$$E: \mathbb{R}^n \rightarrow V, (x^1, \dots, x^n) \mapsto \sum_{i=1}^n x^i E_i.$$

Show that (V, E^{-1}) is a chart for V ; in particular, with the topology defined above, V is thus a topological n -manifold.

- (b) Given a different basis $\tilde{E}_1, \dots, \tilde{E}_n$ for V , show that the charts (V, E^{-1}) and $(V, \tilde{E}^{-1})$ are smoothly compatible.

The collection of all such charts for V defines a smooth structure, called the *standard smooth structure on V* .

Solution: Denote by $\|\bullet\|_V : V \rightarrow \mathbb{R}_{\geq 0}$ the chosen norm on V , and by $\|\bullet\|_{\mathbb{R}^n}$ the standard Euclidean norm on $\mathbb{R}^n$.

- (a) It suffices to show that E is a homeomorphism. First, observe that E is bijective. Now, note that the map $\|\cdot\|' : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ given by $\|x\|' := \|E(x)\|_V$ is a norm on $\mathbb{R}^n$. As all norms on $\mathbb{R}^n$ are equivalent, there exists a constant $c > 1$ such that

$$\frac{1}{c} \|x\|_{\mathbb{R}^n} \leq \|x\|' \leq c \|x\|_{\mathbb{R}^n} \quad \text{for all } x \in \mathbb{R}^n.$$

In particular, both E and E^{-1} are Lipschitz-continuous, and thus E is a homeomorphism.

(b) There exists an invertible matrix $A = (A_i^j)_{1 \leq i, j \leq n} \in \text{GL}(n, \mathbb{R})$ such that $E_i = \sum_j A_i^j \tilde{E}_j$ for each i . Thus, the transition map $\tilde{E}^{-1} \circ E$ is given by

$$(\tilde{E}^{-1} \circ E)(x) = \tilde{E}^{-1}\left(\sum_i x^i E_i\right) = \sum_i x^i \tilde{E}^{-1}(E_i) = \sum_i x^i A_i^j.$$

Hence, the transition map is an invertible linear map, and thus a diffeomorphism (the partial derivatives of the first order are given by constant maps corresponding to the entries of A , and the partial derivatives of higher order vanish).

Exercise 6: Show that two smooth atlases for a topological manifold determine the same smooth structure if and only if their union is again a smooth atlas.

Solution: Suppose A and B are two atlases determining the same smooth structure. By assumption, they are both contained in the same maximal smooth atlas, say C . Consequently, every chart in A must be smoothly compatible with every chart in B , as they both belong to C , which is an atlas. Thus, $A \cup B$ is a collection of charts covering the topological manifold in question, with any given two charts being smoothly compatible, so it is a smooth atlas.

Conversely, suppose $A \cup B$ is a smooth atlas. Then we may find a maximal atlas C containing $A \cup B$. Now the maximal atlas $A_{\max}$ containing A must contain $A \cup B$ by construction, and thus it contains and is equal to C by maximality of C . A similar conclusion can be reached with the maximal atlas $B_{\max}$, so we have $A_{\max} = C = B_{\max}$, hence A and B determine the same smooth structure.