Differential Geometry II - Smooth Manifolds
Winter Term 2026/2027
Lecturer: Dimitri Stelio Wyss
Assistant: Eric Yen-Yo Chen

Exercise Sheet 1

Exercise 1:
Examine which of the following spaces (endowed with the subspace topology) is locally Euclidean:

(a) The closed interval $[0, 1] \subseteq \mathbb{R}$.

(b) The "bent line" $\big\{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ xy = 0\big\} \subseteq \mathbb{R}^2$.

Exercise 2:

(a) *The line with two origins*: Consider the set

$$X = \big\{(x, y) \in \mathbb{R}^2 \mid y \in \{-1, 1\}\big\} \subseteq \mathbb{R}^2$$

and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff.

(b) Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Exercise 3: Consider the subset

$$V = \big\{(x, y) \in \mathbb{R}^2 \mid (x - 1)(x - y) = 0\big\} \subseteq \mathbb{R}^2$$

endowed with the subspace topology. Show that V is not a topological manifold.

Exercise 4 (*Product manifolds*)**:**
Let $M_1, \ldots, M_k$ be topological manifolds of dimensions $n_1, \ldots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$.

Exercise 5 (*Finite-dimensional vector spaces*)**:**
Let V be an $\mathbb{R}$-vector space of dimension n. Recall that any norm on V determines a topology, which is independent of the choice of norm. Show that V has a natural smooth manifold structure as follows:

(a) Pick a basis $E_1, \dots, E_n$ for V and consider the map

$$E\colon \mathbb{R}^n \to V,\ (x^1, \dots, x^n) \mapsto \sum_{i=1}^{n} x^i E_i.$$

Show that (V, E^{-1}) is a chart for V; in particular, with the topology defined above, V is thus a topological n-manifold.

(b) Given a different basis $\widetilde{E}_1, \dots, \widetilde{E}_n$ for V, show that the charts (V, E^{-1}) and $(V, \widetilde{E}^{-1})$ are smoothly compatible.

The collection of all such charts for V defines a smooth structure, called the *standard smooth structure on V*.

Exercise 6: Show that two smooth atlantes for a topological manifold determine the same smooth structure if and only if their union is again a smooth atlas.