Stability of collisionless systems

3rd part

Outlines

The stability of uniformly rotating systems

The stability of rotating disks : spiral structures

- Spirals properties
- The dispersion relation for a razor thin fluid disk

The origin of spiral structures: another view

Vertical instabilities : Nature is always more tricky…

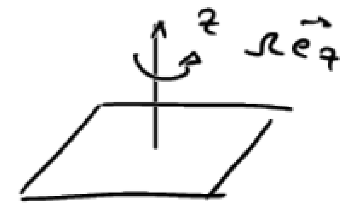
The stability of uniformly rotating systems

The stability of uniformely rotating systems

- Flattened systems : the geometry is more complex
- Reservoir of kinetic energy (rotation) to feed unstable modes

The uniformly rotating sheet

- infinite disk of zero thickness with surface density Σ_0
  - plane $z = 0$
  - rotation $\vec{\Omega} = \Omega \vec{e}_z$



- 2D perturbation / evolution (no warp, no bending)

Equations in the rotating frame $\Sigma_d(x,y),\ \phi_d(x,y),\ P(x,y),\ \vec{v}(x,y)$

5 unknown
5 equations

① Continuity equation

$$\frac{\partial \Sigma}{\partial t} + \vec{\nabla}(\Sigma\ \vec{v}) = 0$$

② Euler Equation

Coriolis force — Centrifugal force

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\Sigma_d} - \vec{\nabla}\phi - 2\Omega(v_x \vec{e}_x + v_y \vec{e}_y) + \Omega^2(x\vec{e}_x + y\vec{e}_y)$$

③ Poisson equation

$$\nabla^2\phi = 4\pi G\Sigma\,\delta(z)$$

④ Equation of state (barotropic)

$$p(x,y) = p(\Sigma(x,y))$$

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla}\phi_e$

$$\Sigma_{d0} \rightarrow \Sigma_{d0} + \varepsilon \Sigma_{d1}(x, y, t)$$

$$\vec{V}_0 = 0 \rightarrow \varepsilon V_1(x, y, t)$$

$$\Sigma_0 \rightarrow \Sigma_0 + \varepsilon \Sigma_1(x, y, t)$$ total density

$$\phi_0 \rightarrow \phi_0 + \varepsilon \phi_1(x, y, t)$$ total potential

$$= \varepsilon \phi_{d1}(x, y, t) + \varepsilon \phi_e(x, y, t)$$

A first order in ε, we get

① Continuity equation

$$\frac{\partial}{\partial t}\Sigma_{d1} + \Sigma_0 \vec{\nabla}\vec{V}_1 = 0$$

② Euler Equation

$$\frac{\partial}{\partial t}\vec{V}_1 = -\frac{V_s^2}{\Sigma_0}\vec{\nabla}\Sigma_{d1} - \vec{\nabla}\phi_1 - 2\vec{\Omega}\times\vec{V}_1$$

contribution of the rotation

③ Poisson equation

$$\nabla^2\phi_1 = 4\pi G \Sigma_1 \delta(z)$$

the rest is similar to the homogeneous case

④ "Equation of state"

$$V_s^2 = \left.\frac{\partial p}{\partial \Sigma}\right|_{\Sigma_0}$$

Solutions of the form

$$\Sigma_{d\lambda}(x,y,t) \sim e^{i(\vec{k}\vec{x} - \omega t)}$$

leads to : (without the perturbation)

$$\boxed{\omega^2 = v_s^2 k^2 - 2\pi G|k|\Sigma_0 + 4\Omega^2}$$

the rotation helps to stabilize

(Dispersion relation for an homogeneous fluid

$$\boxed{\omega^2 = v_s^2 k^2 - 4\pi G \rho_0}$$
)

Dispersion relation for a uniformly rotating stellar sheet

- if the velocity DF is Maxwellian

$$f_0(\vec{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} \, e^{-\frac{v^2}{2\sigma^2}}$$

($\vec{v}$: in the plane)

$$\boxed{\omega^2 = \sigma^2 k^2 - 2\pi G |k| \Sigma_0 + 4\Omega^2}$$

The stability of rotating disks:

spiral structures

Spiral galaxies : disky structures

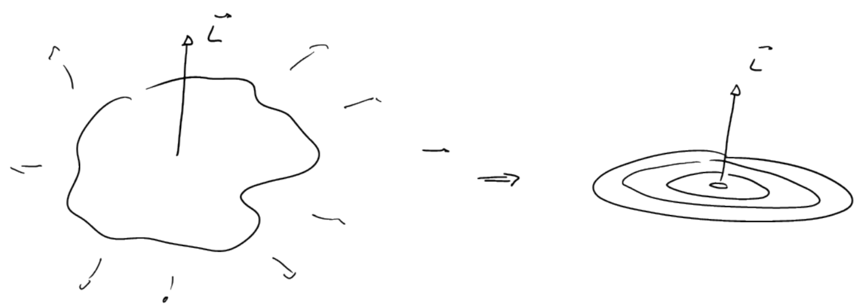
M83

NGC4945

Questions:

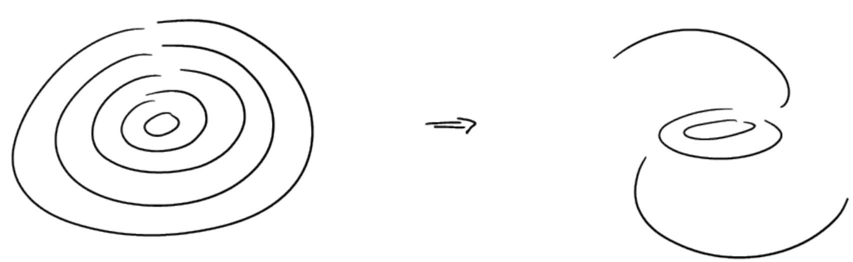
Why galaxies are disks ?

- Gas radiates energy but not angular momentum.
- Circular orbits have a minimum energy for a given angular momentum.
- For a given angular momentum distribution, the state of the lowest energy is a flat disk.



Why disk galaxies display complex structures : bars, spirals ?

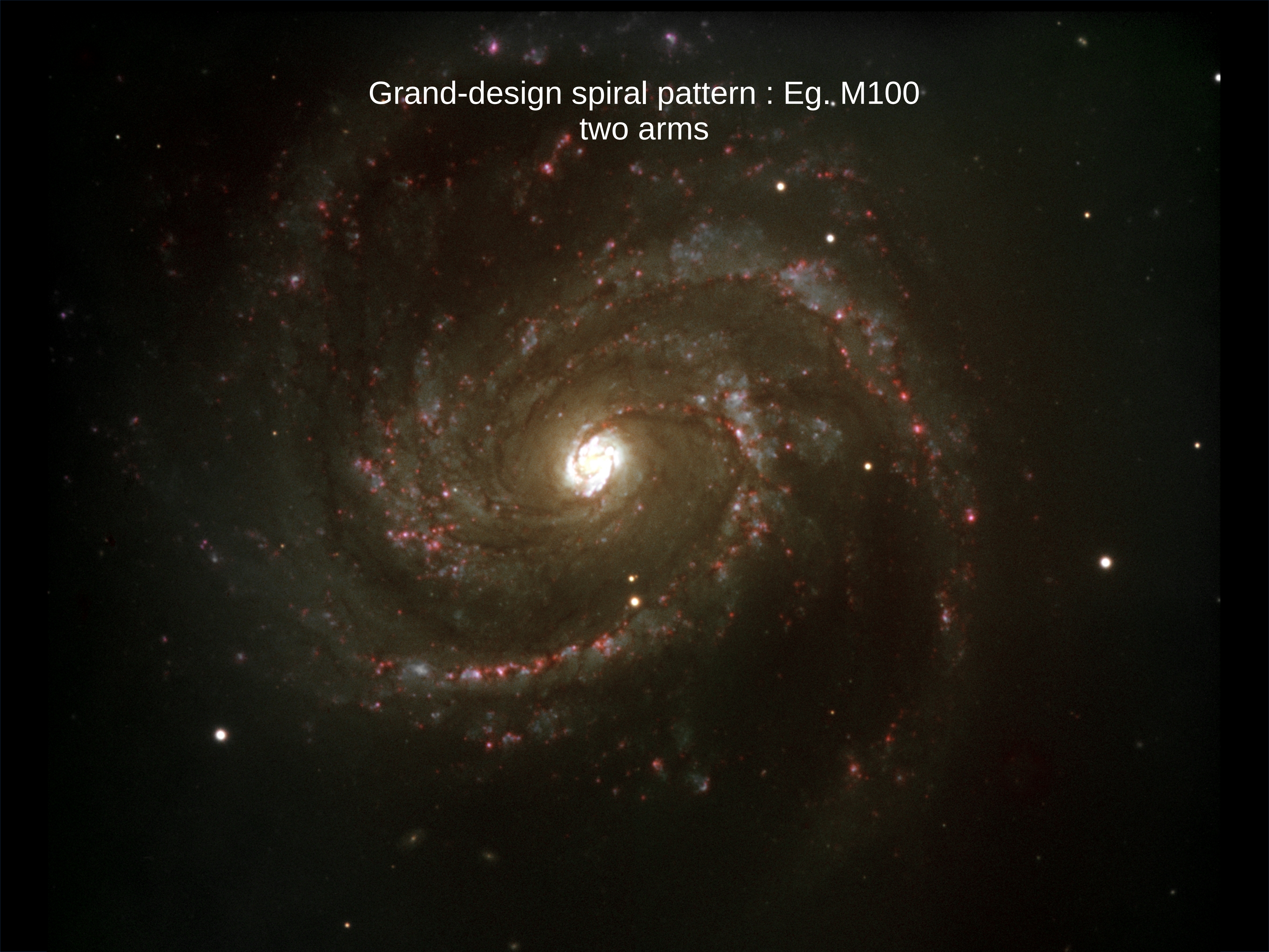
- Dynamically cool systems (low velocity dispersion) are strongly unstable
- Further cooling requires to avoid the constraints provided by the angular momentum conservation : need to break the symmetry !



Properties of spirals:

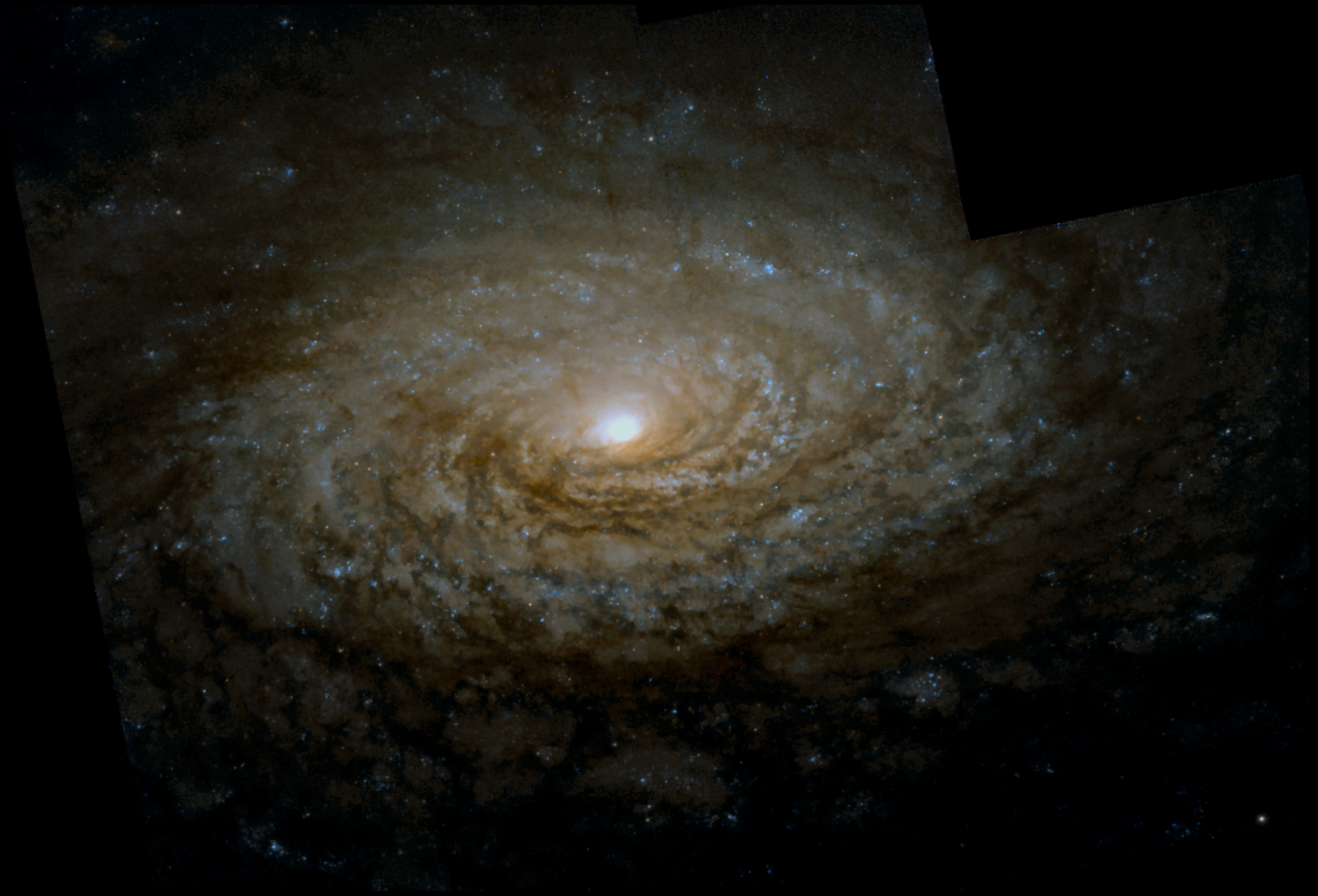
Different spiral patterns:

- Grand-design
- Intermediate-scale
- Flocculent


Grand-design spiral pattern : Eg. M100
two arms

Intermediate-scale spiral pattern : Eg. M101

Flocculent spirals : Eg. M63



Properties of spirals:

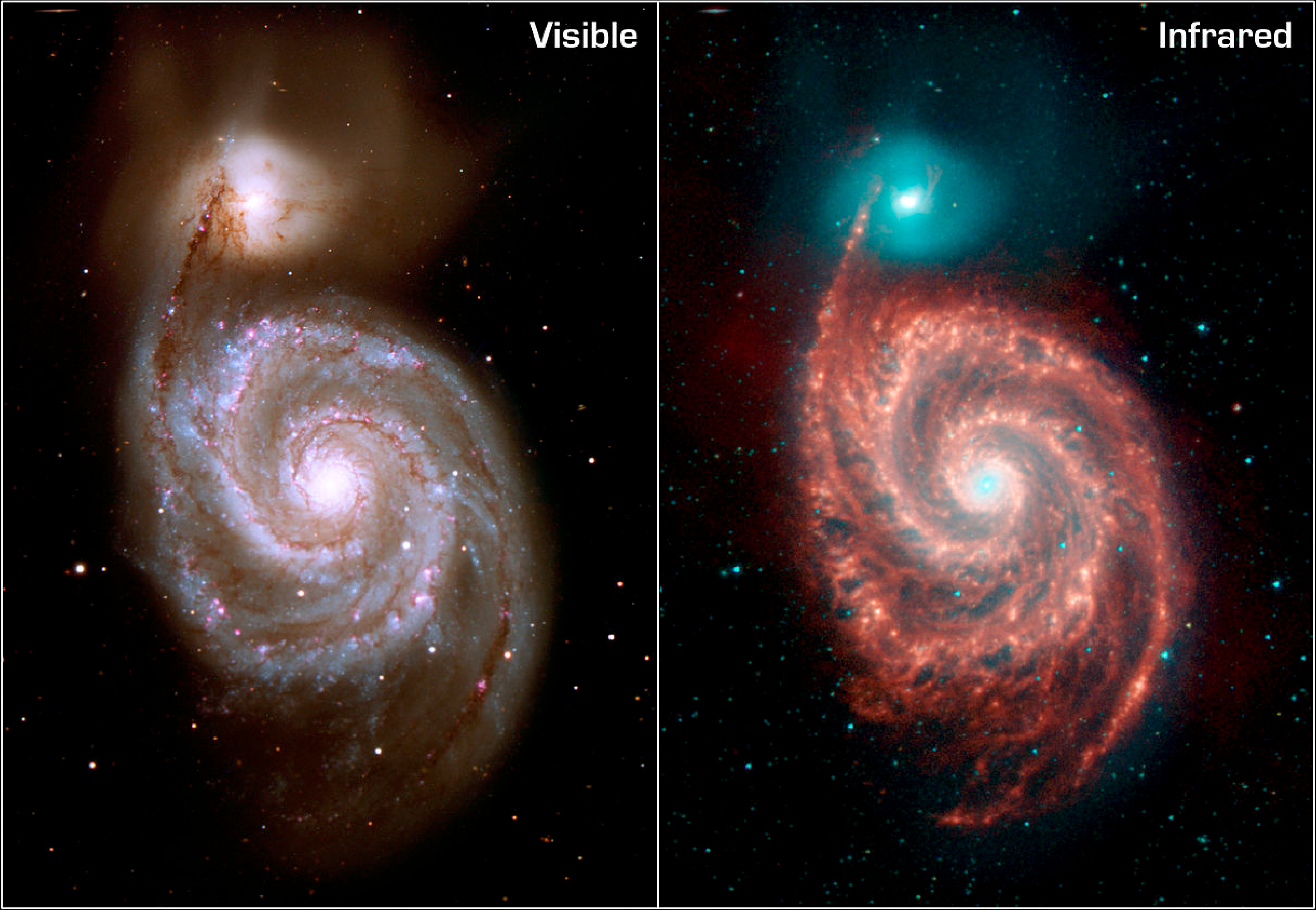
Different spiral patterns:

- Grand-design
- Intermediate-scale
- Flocculent

The bulk of the matter participates to the spirals

- Coherence in different wavelengths tracing different components

Grand-design spiral pattern

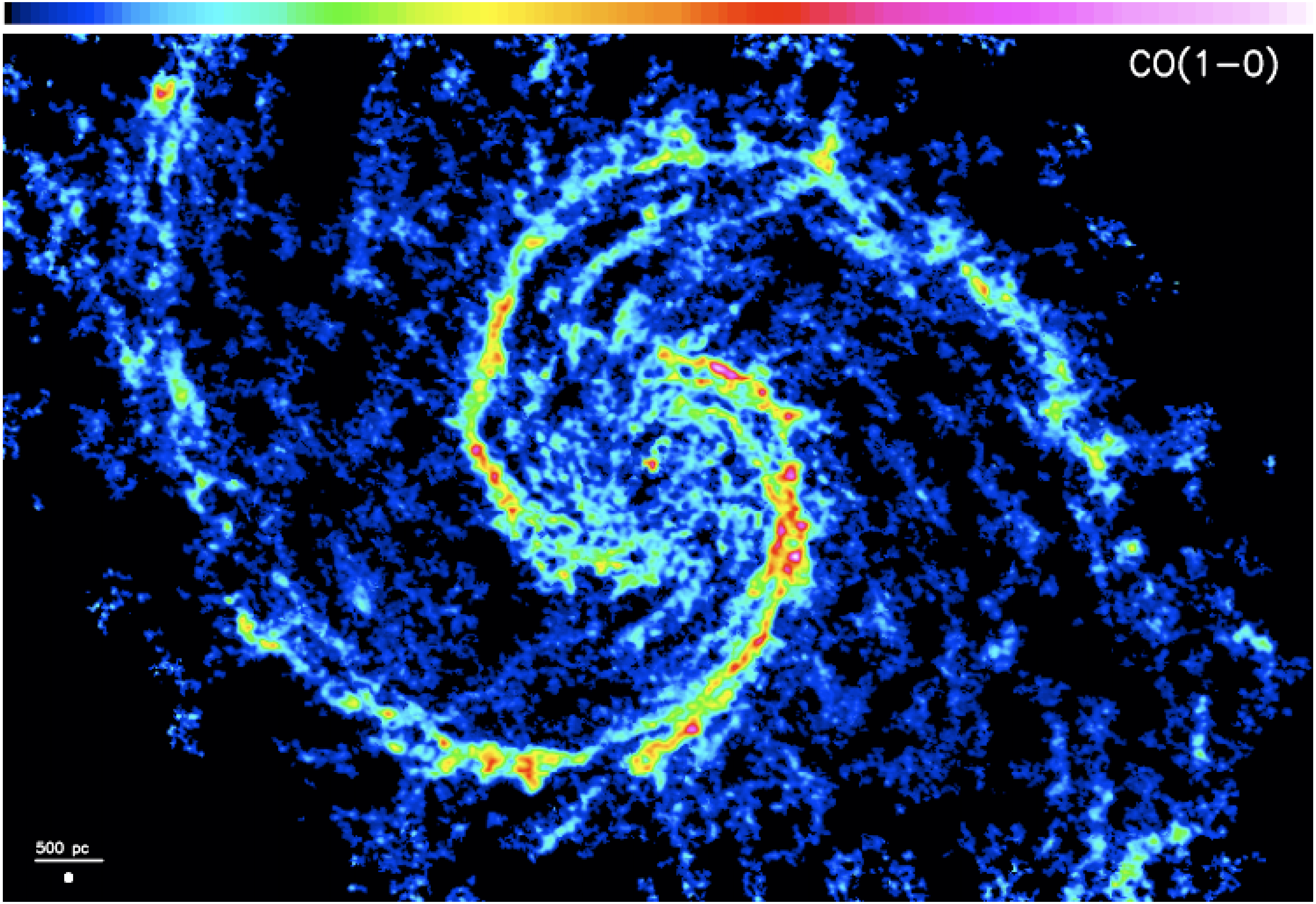


Spiral Galaxy M51 (“Whirlpool Galaxy”)

Spitzer Space Telescope • IRAC

NASA / JPL-Caltech / R. Kennicutt (Univ. of Arizona)

ssc2004-19a



Schinnerer et al. 2013

Model spiral arms

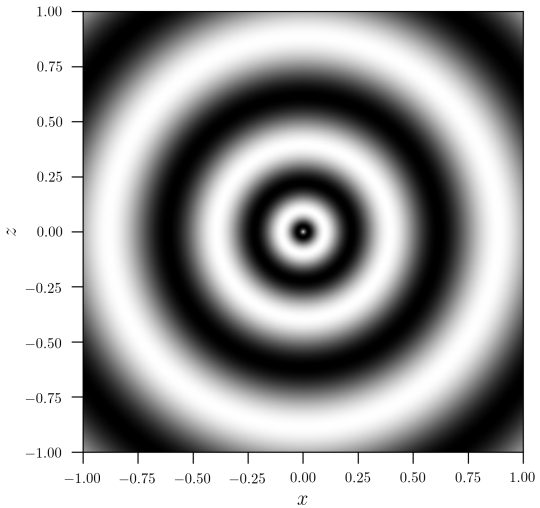
Surface density of a spiral galaxy

shape function

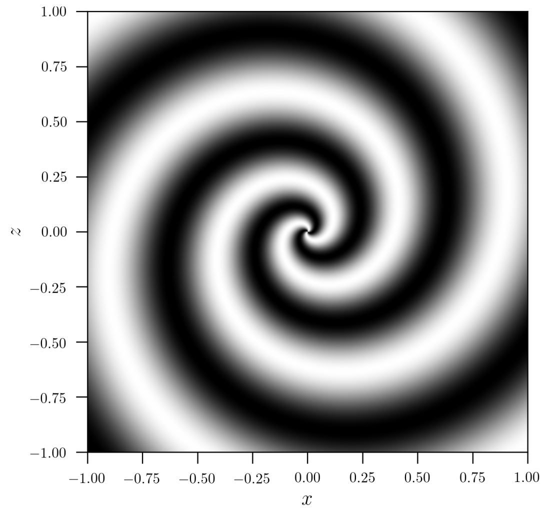
$$\Sigma(R,\phi,t) = Re\left[H(R,t)e^{i(m\phi+f(R,t)-\omega t)}\right]$$

arms

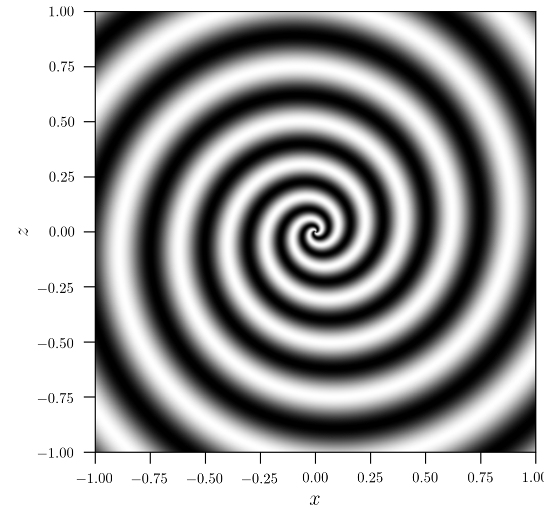
$m = 0, f = 20R^{1/2}$



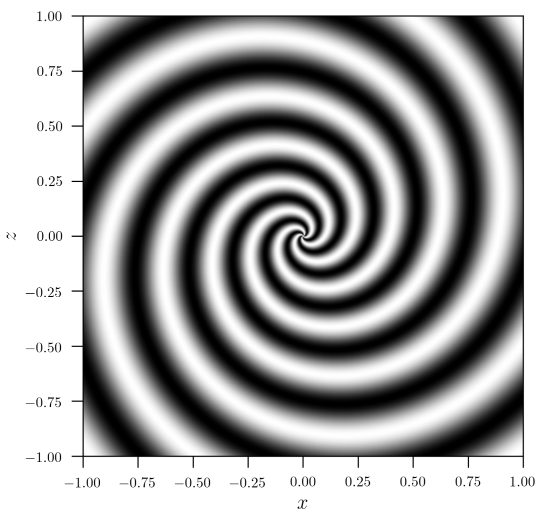
$m = 2, f = 20R^{1/2}$



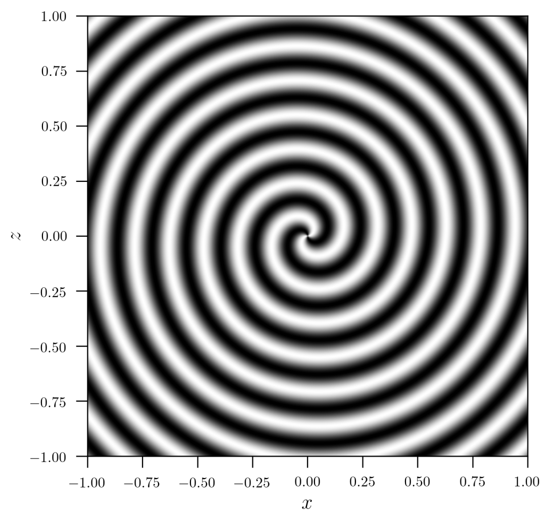
$m = 2, f = 40R^{1/2}$



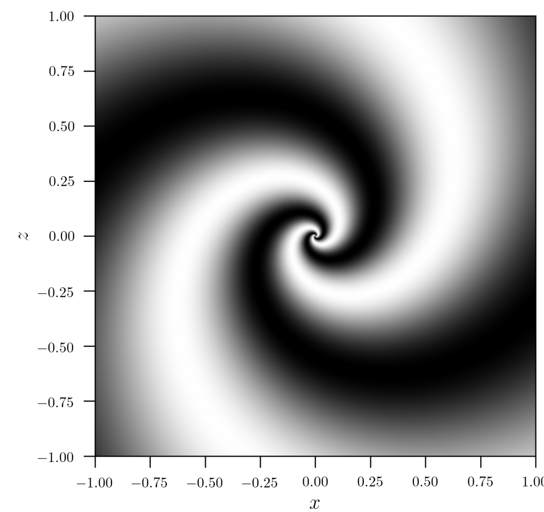
$m = 4, f = 40R^{1/2}$



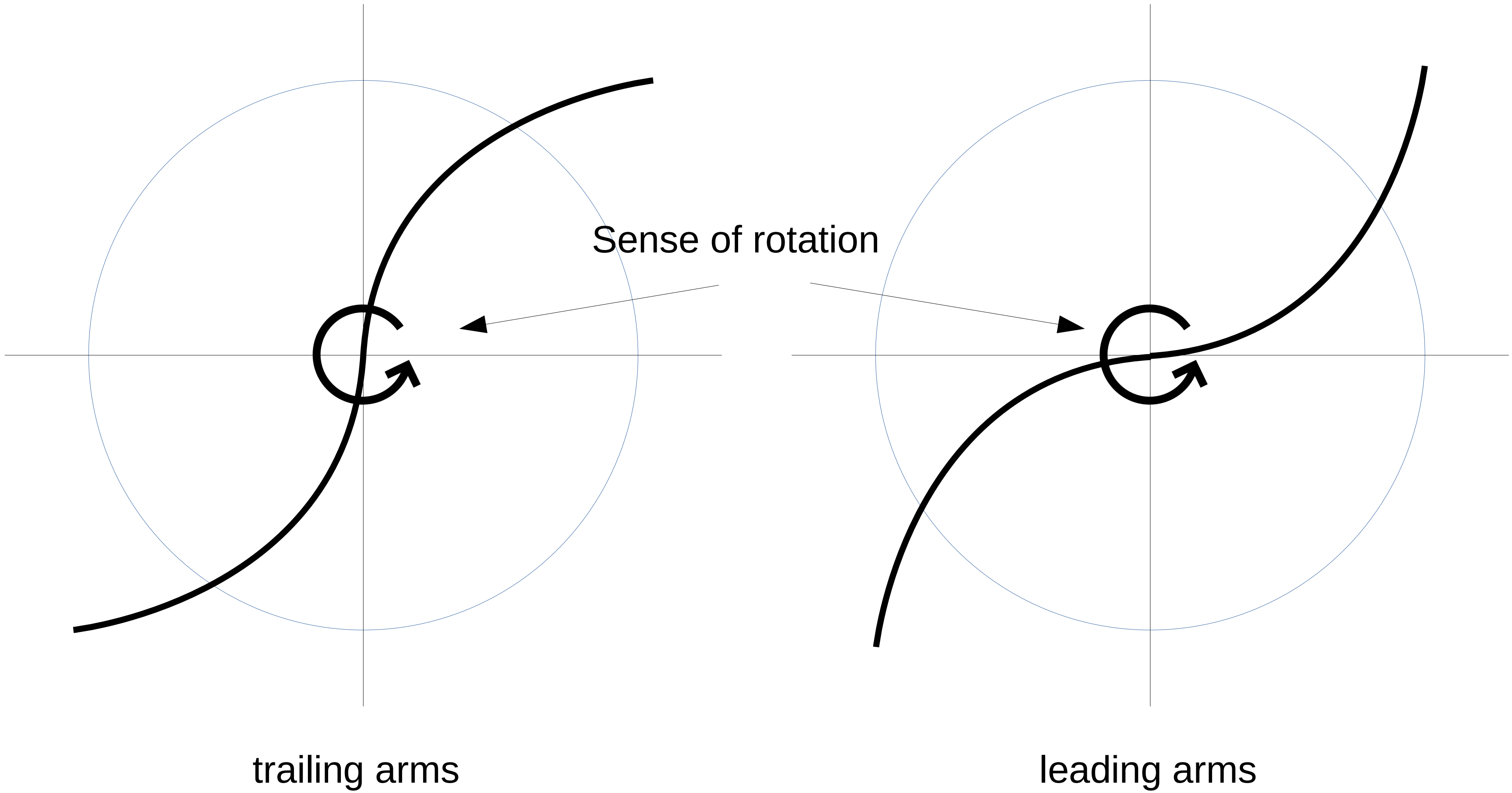
$m = 2, f = 40R$



$m = 2, f = 40R^{0.1}$



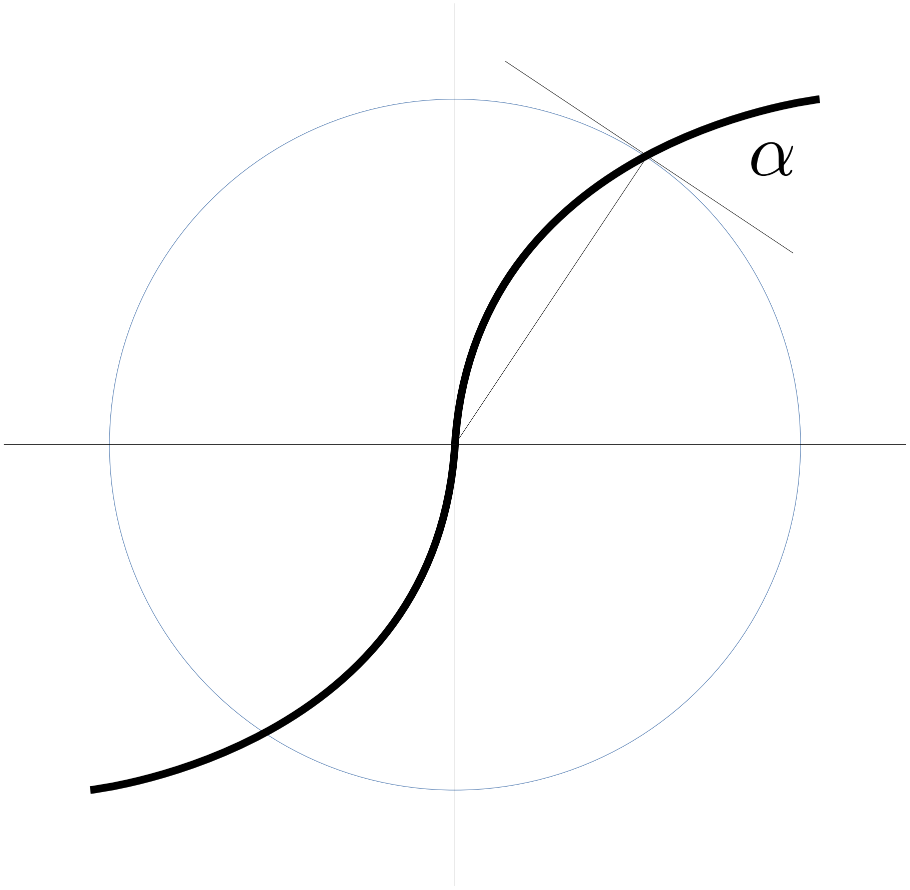
Leading and trailing spirals:



The majority of spiral arms are trailing !

Pitch angle

<u>Definition</u>: pitch angle : angle between the tangent of a circle and the spiral



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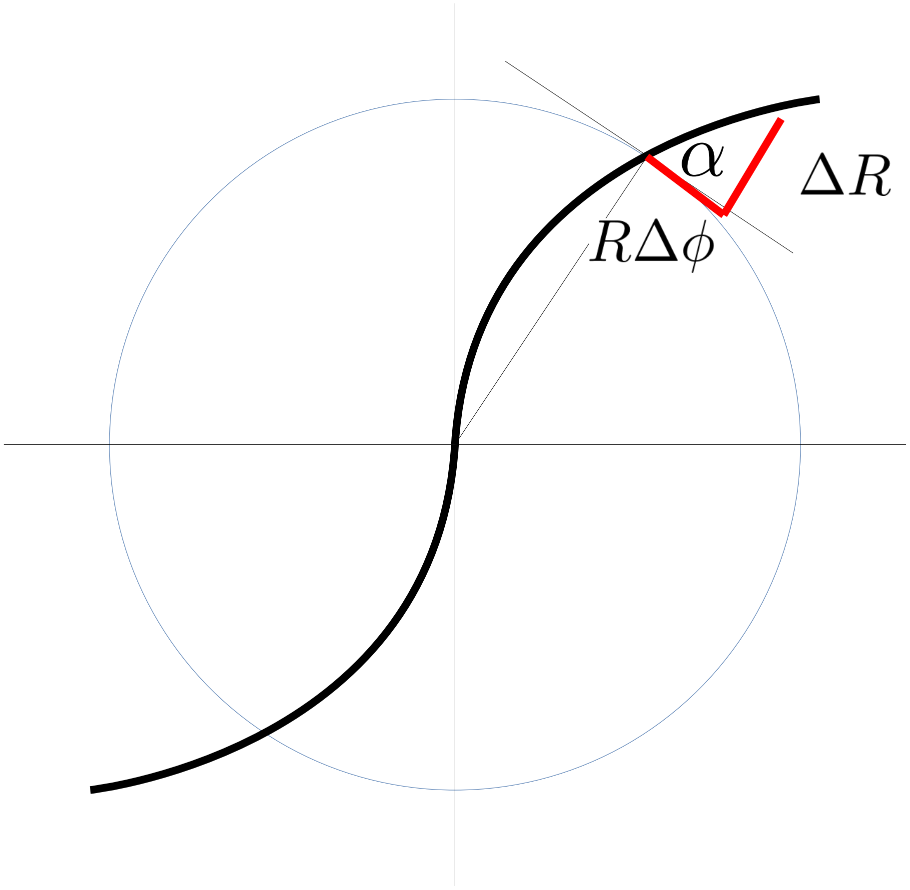
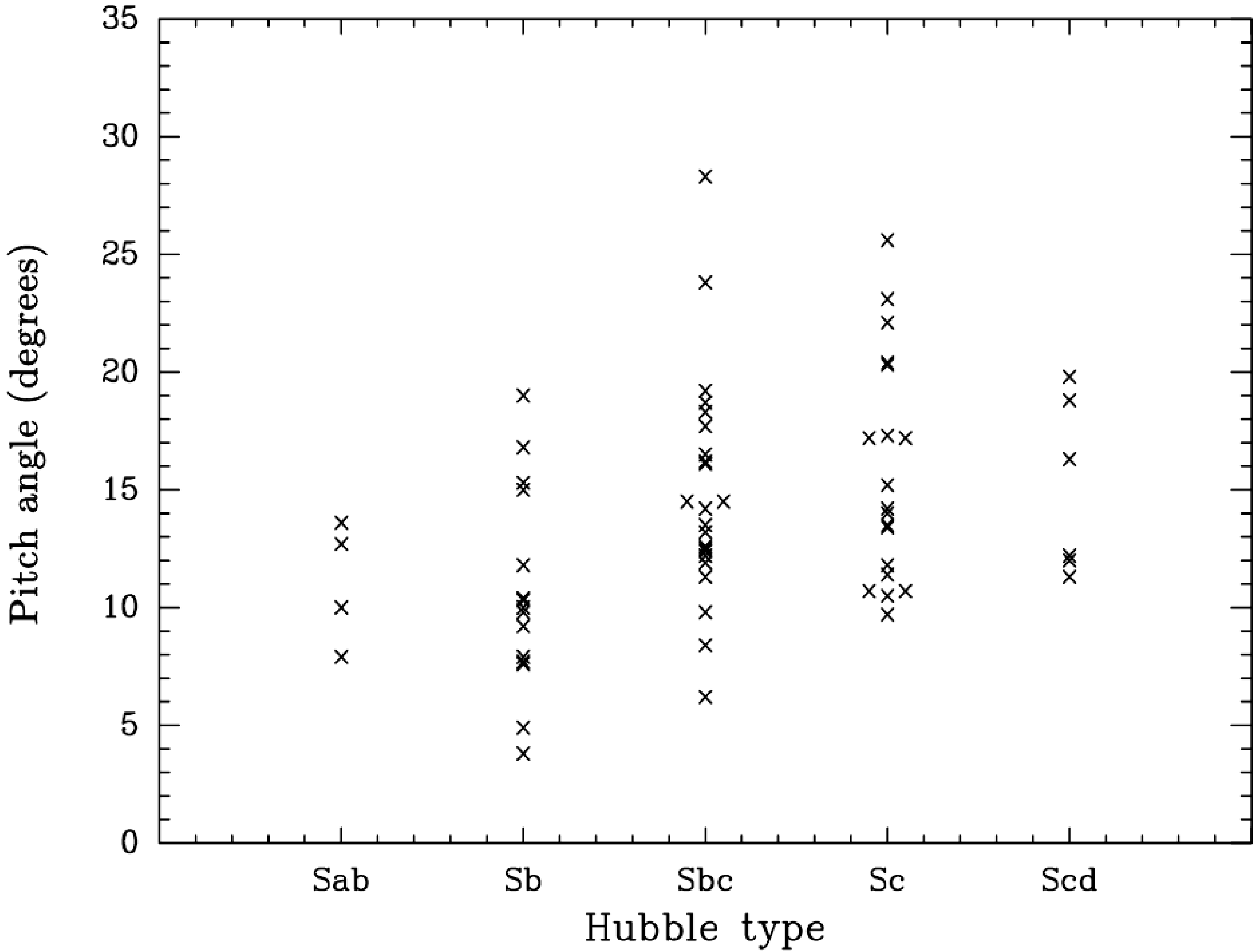


$$\tan\alpha = \frac{\Delta R}{R\Delta\phi}$$

Pitch angle as a function of the Hubble type (Ma 2002)



Origin of the spiral structure:

differential rotation ?

Winding problem

> Are observed spirals the result of differential rotation?

Assume a constant velocity curve

$$V(R) = R\,\Omega(R) \,\hat{=}\, 200\,\mathrm{km/s} = V_0$$

$$\Omega(R) = \frac{V_0}{R}$$

$$\phi(R,t) = \Omega(R)\,t + \phi_0 = \frac{V_0}{R}\,t + \phi_0$$

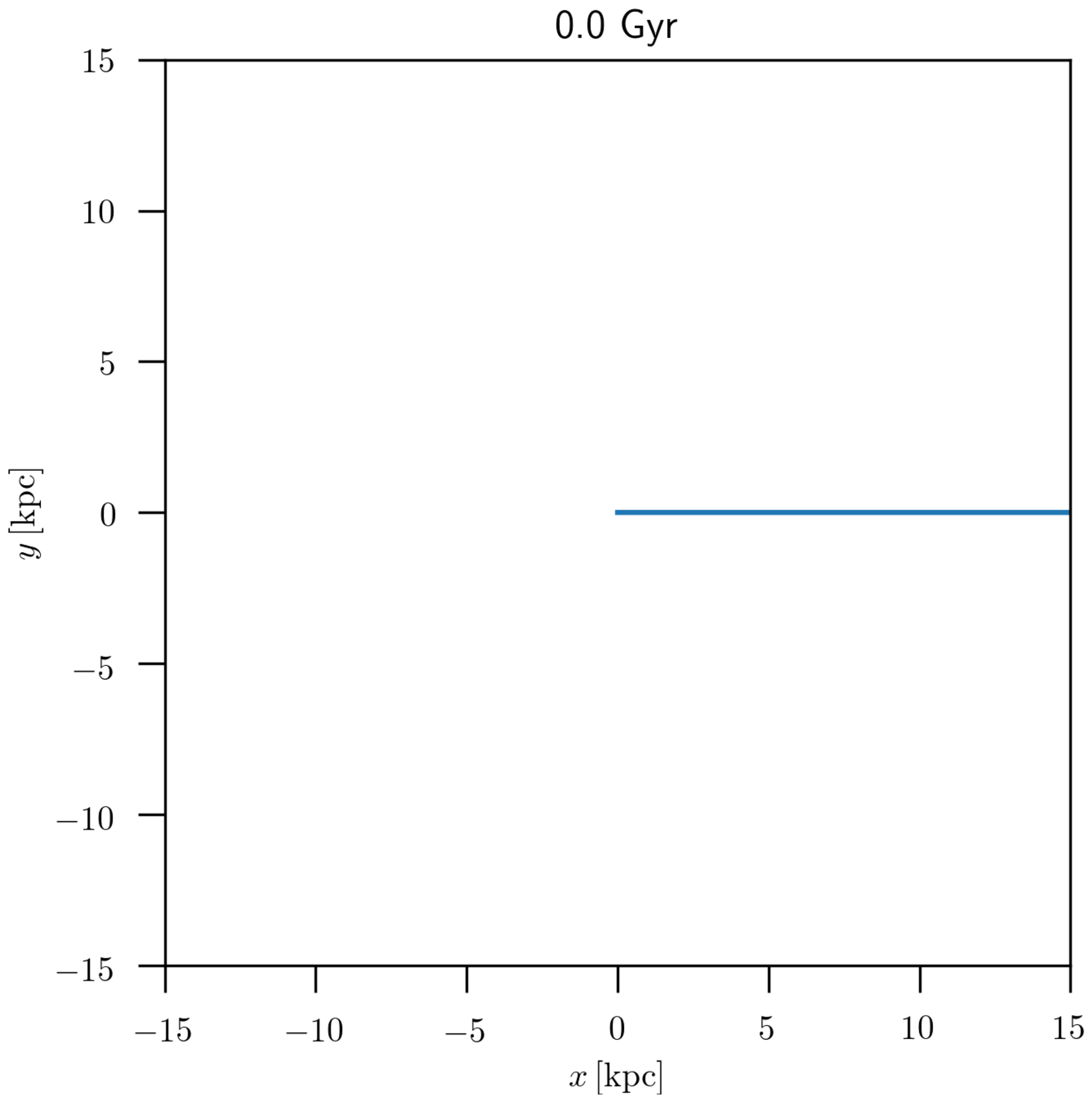
$$\frac{\partial\phi}{\partial R} = -\frac{V_0 t}{R^2} \Rightarrow \frac{\Delta R}{R\,\Delta\phi} = \frac{R}{t V_0} \Rightarrow \alpha = \arctan\left(\frac{R}{t V_0}\right)$$

for $t = 10\,\mathrm{Gyr}$, $R = 5\,\mathrm{kpc}$, $V_0 = 200\,\mathrm{km/s}$

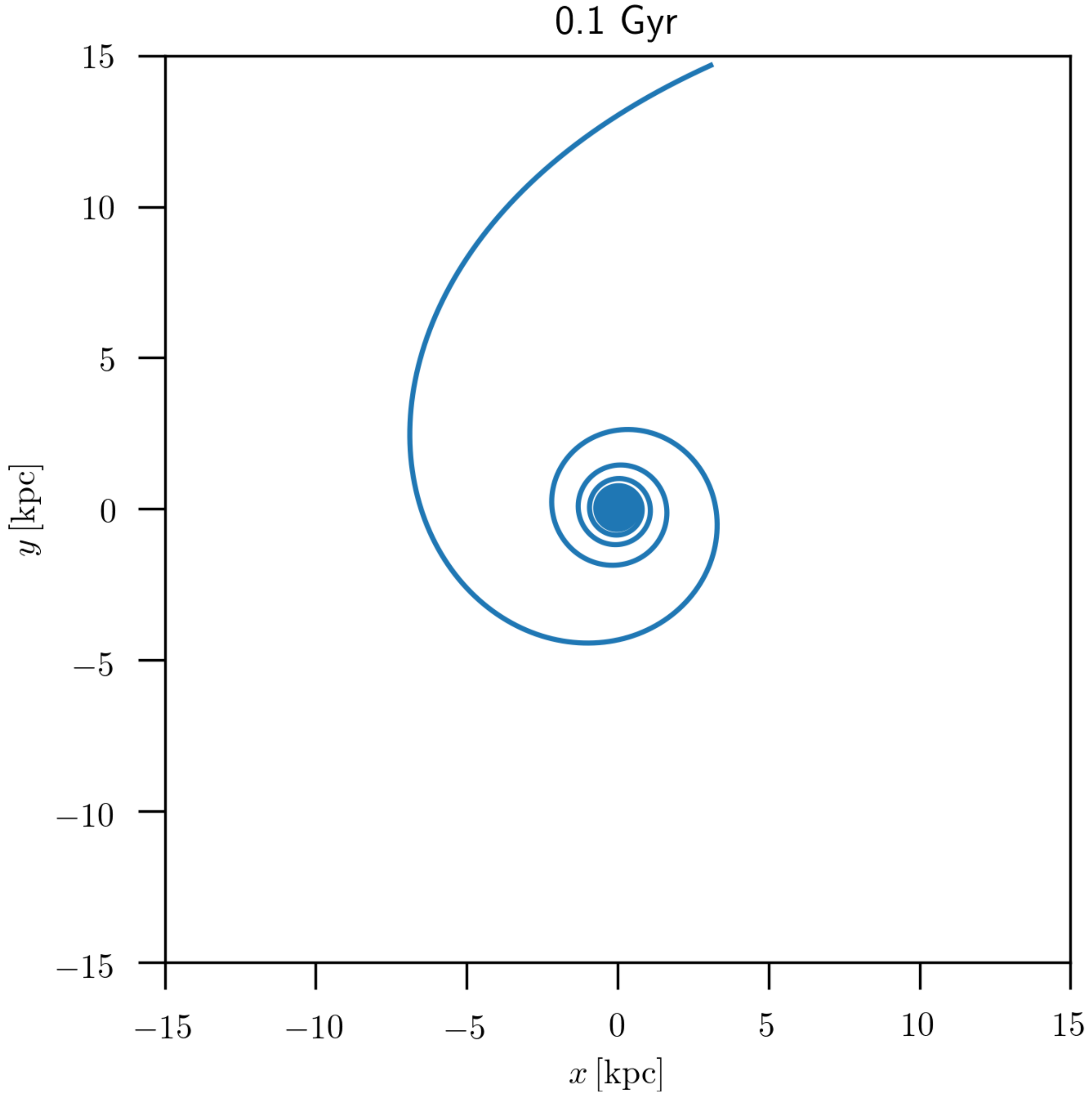
$$\boxed{\alpha = 0.14^\circ} \ll 10^\circ - 15^\circ$$

⚠

Winding problem : V=200 km/s



Winding problem : V=200 km/s



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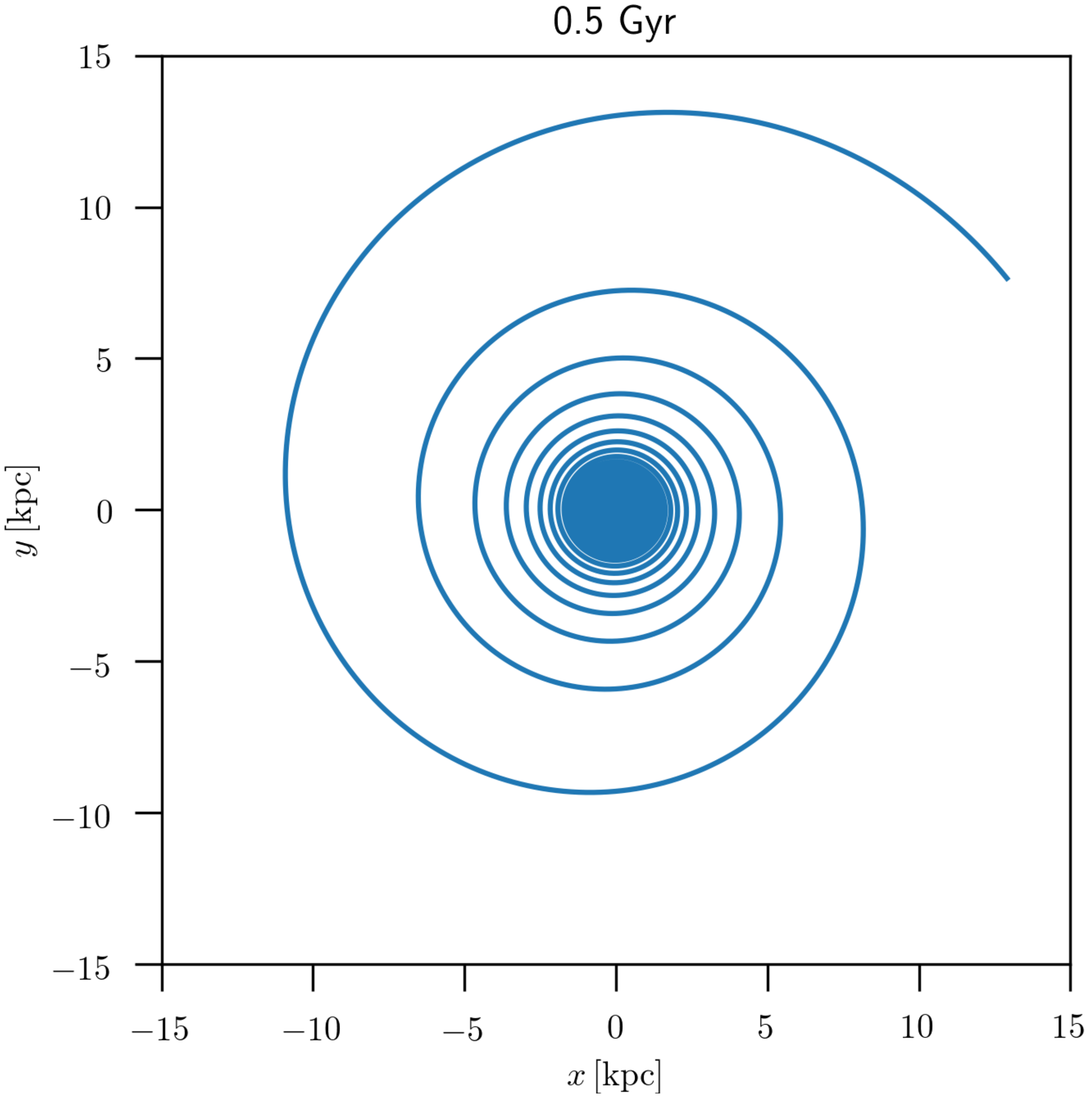
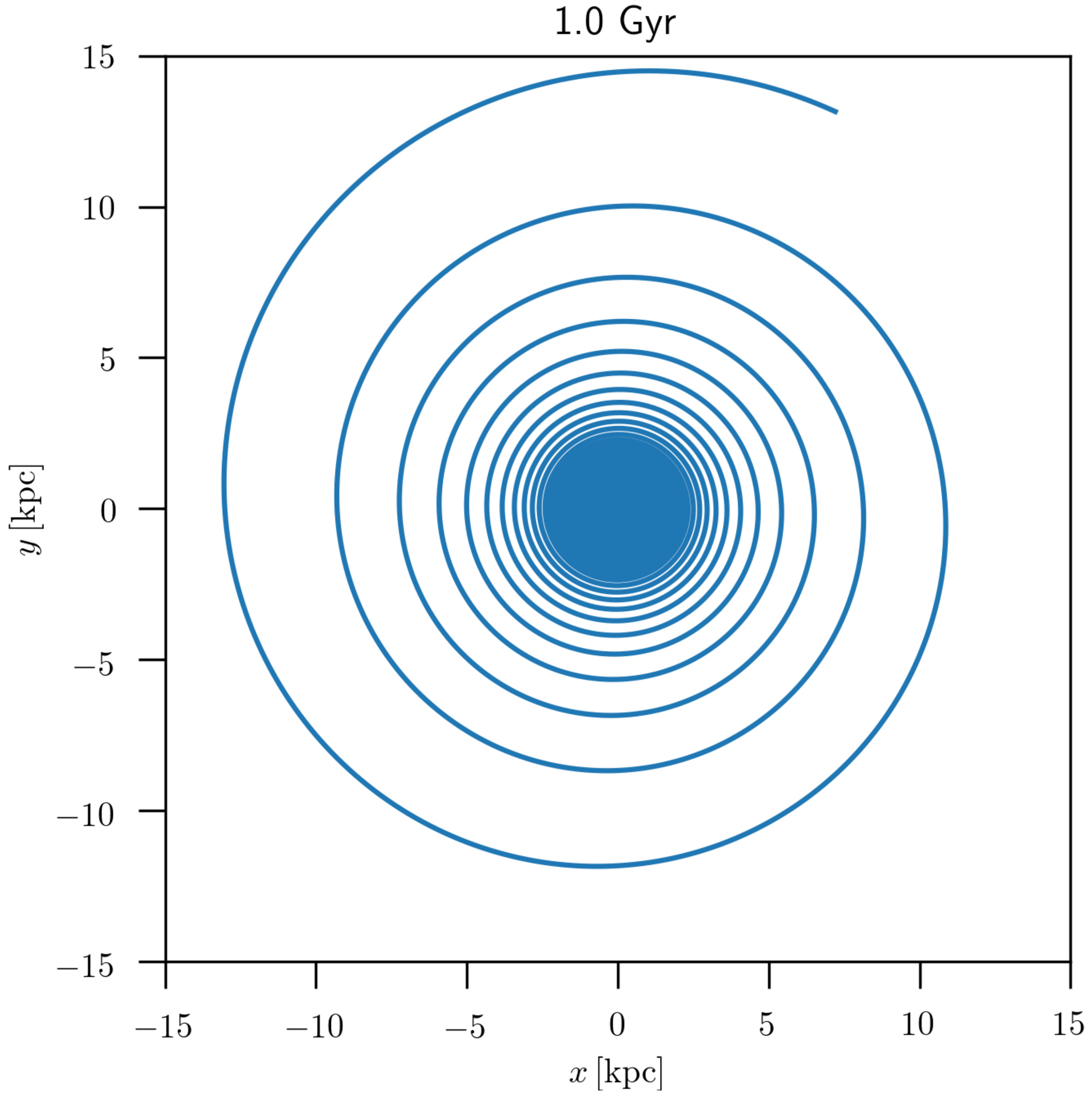
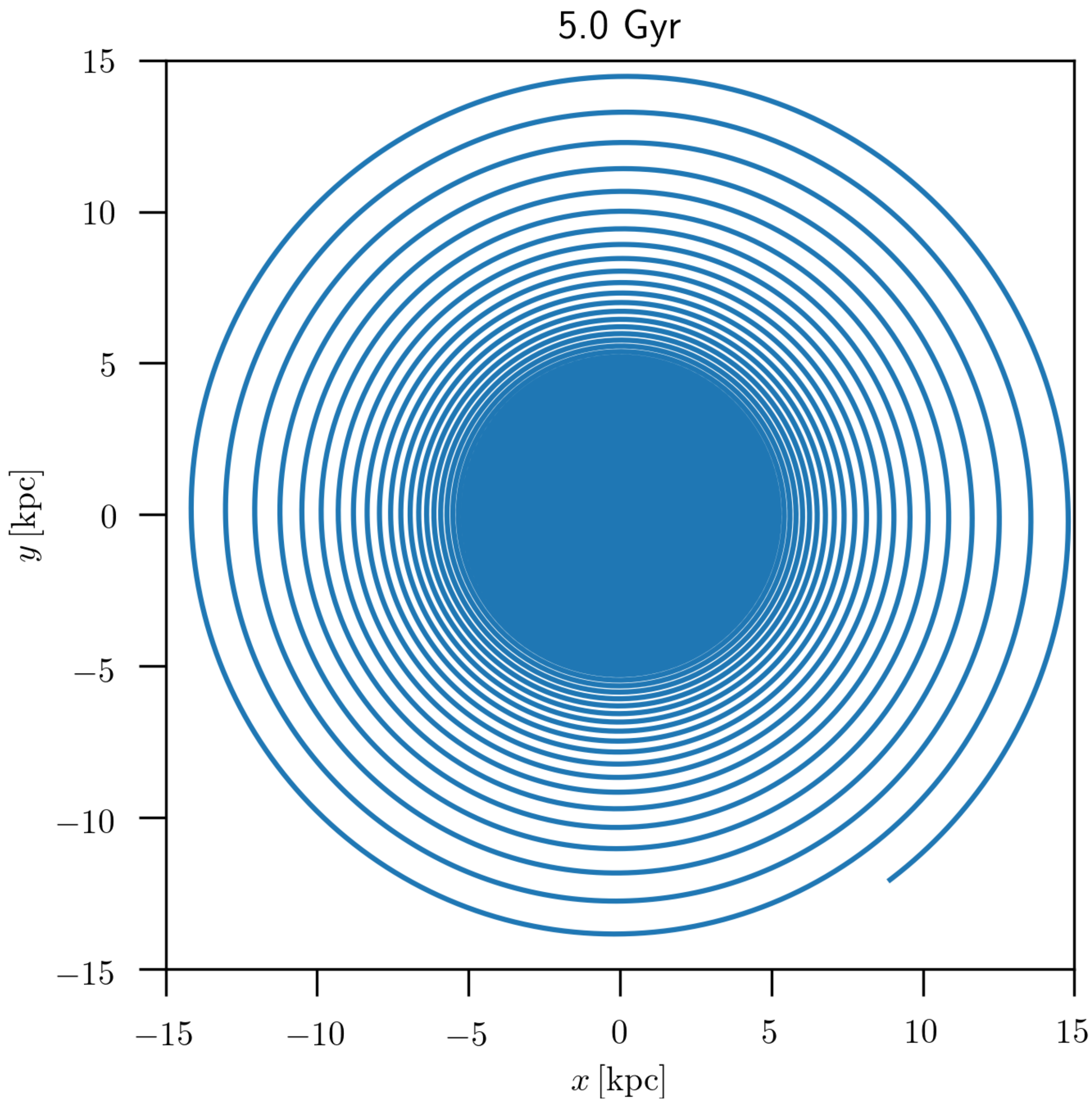


Winding problem : V=200 km/s



Winding problem : V=200 km/s



Possible solutions to the winding problem

1. A spiral arm is <u>a transient phenomena</u>, but the spiral pattern is statistically in a steady state.

   <u>Example</u> : a spiral arm traces young stars, until they die off
   $\rightarrow$ α could be much bigger

   Could explain flocculent galaxies, but not grand-design ones

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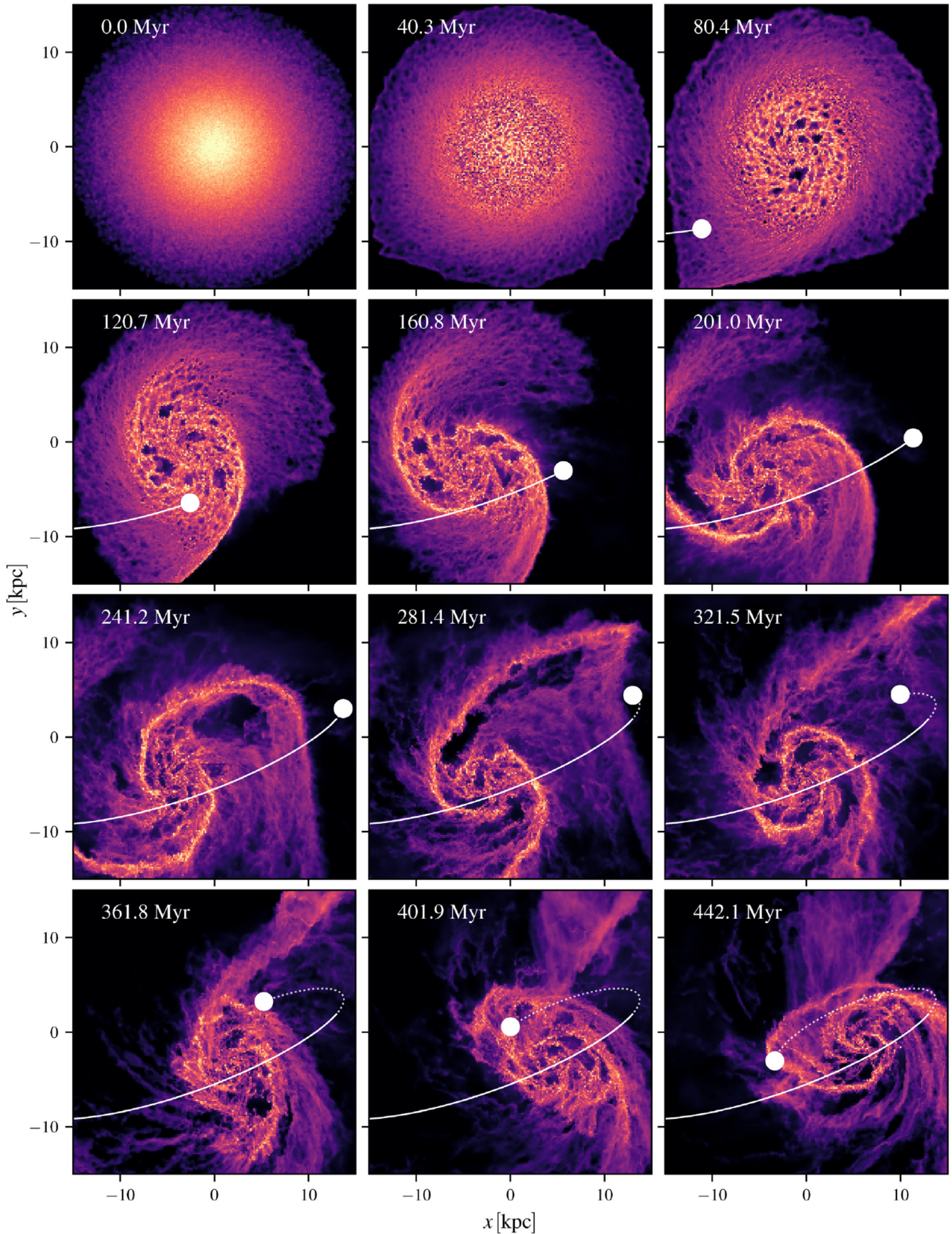
2. The spiral pattern is <u>a temporary phenomena</u>, resulting for example from a recent event, like a merger or an interaction.


0.0 Myr
40.3 Myr
80.4 Myr
120.7 Myr
160.8 Myr
201.0 Myr
241.2 Myr
281.4 Myr
321.5 Myr
361.8 Myr
401.9 Myr
442.1 Myr
y [kpc]
x [kpc]
10
0
−10

Tress 2020

Possible solutions to the winding problem

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2. The spiral pattern is <u>a temporary phenomena</u>, resulting for example from a recent event, like a merger or an interaction.

3. The spiral structure is <u>a stationary density wave</u> that rotate rigidly in ρ and ϕ and thus, not subject to the winding problem (Lin-Shu 1964).
   The spiral pattern is a kind a mode, like a drum that vibrate.

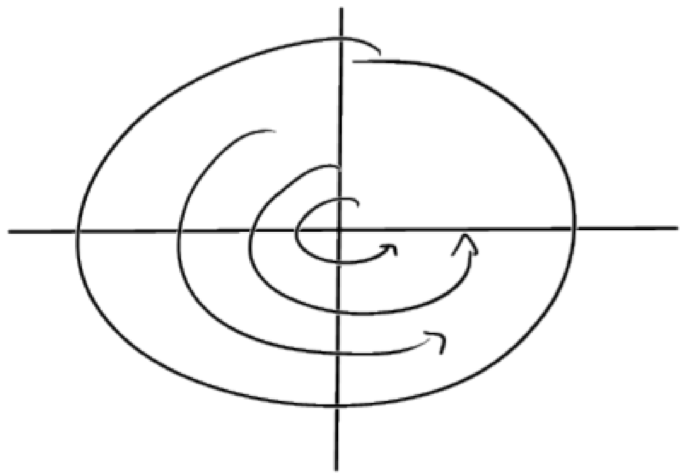
The stability of rotating disks :

the dispersion relation (in a nutshell)

The dispersion relation for a razor thin rotating fluid disk

Polar coordinates in the inertial rest frame

$$\Sigma_d(R,\phi)\,,\quad \Phi_d(R,\phi)\,,\quad p(R,\phi)\,,\quad \vec{v}(R,\phi) = v_R(R,\phi)\,\vec{e}_R + v_\phi(R,\phi)\,\vec{e}_\phi$$



the disc can have a differential rotation

① Continuity equation $\quad \dfrac{\partial \Sigma_d}{\partial t} + \vec{\nabla}(\Sigma_d \vec{v}) = 0$

$$\boxed{\frac{\partial \Sigma_d}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\,\Sigma_d\, v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\Sigma_d\, v_\phi) = 0}$$

② Euler equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\Sigma_d} - \vec{\nabla}\phi$$

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial \phi}{\partial R} - \frac{1}{\Sigma_d}\frac{\partial p}{\partial R}$$

$$\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial \phi}{\partial \phi} - \frac{1}{\Sigma_d}\frac{1}{R}\frac{\partial p}{\partial \phi}$$

③ Poisson

$$\nabla^2\phi = 4\pi G \Sigma \delta(z)$$

④ Equation of state (polytropic)

$$p = K \Sigma_d^{\gamma}$$

$$v_s^2 = \gamma K \Sigma_0^{\gamma-1}$$ (unperturbed sound speed)

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla} \phi_e$

$$\Sigma_{d0} \rightarrow \Sigma_{d0} + \varepsilon \Sigma_{d1}$$

$$V_{R_0} = 0 \rightarrow \varepsilon V_{R1}$$

$$V_{\phi_0} \rightarrow V_{\phi_0} + \varepsilon V_{\phi_1}$$

$$h_0 \rightarrow h_0 + \varepsilon h_1$$

$$\phi_{d0} \rightarrow \phi_{d0} + \varepsilon \phi_{d1}$$

disk only will ... the perturbation

$$\phi_0 = \phi_{d0} \rightarrow \phi_{d0} + \underbrace{\varepsilon \phi_1}_{\varepsilon \phi_{d1} + \varepsilon \phi_e}$$

total potential, includes the perturbation

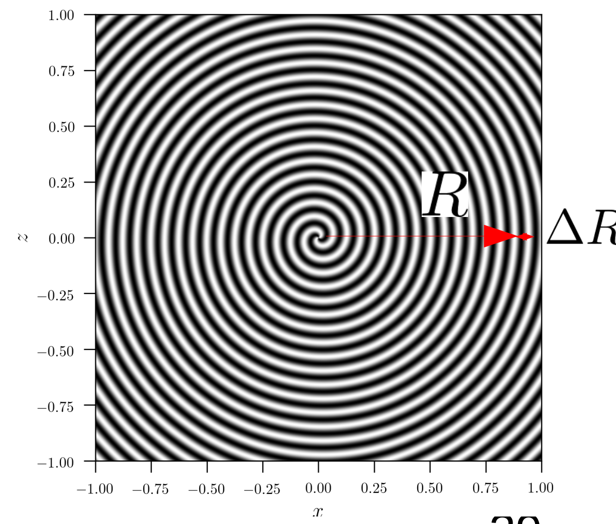
+ solutions of the form

$$\Sigma_{d1}(R, \phi, t) \sim e^{i(m\phi + f(R,t) - \omega t)}$$

+ WKB approximation to solve the Poisson equation

$$\phi_{d1} = -\frac{2\pi G}{|k|} e^{i(m\phi + f(R,t) - \omega t)}$$

$$k = \frac{\partial f}{\partial R}$$



We obtain the dispersion relation: (for $\Sigma_e = 0$)

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

κ : epicycle radial frequency

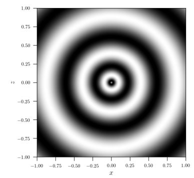
<u>Note:</u> if $\Omega(R) = \text{cte}$ $\kappa = 2\Omega$ and $m = 0$ we recover the dispersion relation for uniformly rotating sheet

$$\omega^2 = 4\Omega^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

Interpretation

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

Axisymmetric perturbations $\quad m = 0$



"Cold disk" $\quad v_s = 0$

$$\omega^2 = \kappa^2 - 2\pi G \Sigma_0 |k|$$

$$k_{crit} := \frac{\kappa^2}{2\pi G \Sigma_0} \qquad \lambda_{crit} := \frac{2\pi}{k_{crit}} = \frac{4\pi^2 G \Sigma_0}{\kappa^2}$$

$$\omega^2 = 2\pi G \Sigma_0 \left(k_{crit} - |k| \right)$$

$|k| > k_{crit} \quad (\lambda < \lambda_{crit}) \quad \omega^2 < 0$ **UNSTABLE**

$|k| < k_{crit} \quad (\lambda > \lambda_{crit}) \quad \omega^2 > 0$ **STABLE**

The differential rotation stabilizes the system at large scale

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

"Non rotating fluid disk"

$\kappa = 0$

⚠ not realistic

$$\omega^2 = -2\pi G \Sigma_0 |k| + v_s^2 k^2$$

$$k_{crit} := \frac{2\pi G \Sigma_0}{v_s^2} \quad (\equiv k_{Jeans}) \qquad \lambda_{crit} = \frac{2\pi}{k_{crit}} = \frac{v_s^2}{G \Sigma_0}$$

$$\omega^2 = v_s^2 |k| (|k| - k_{crit})$$

$|k| > k_{crit}$ ($\lambda < \lambda_{crit}$) $\omega^2 > 0$ **STABLE**

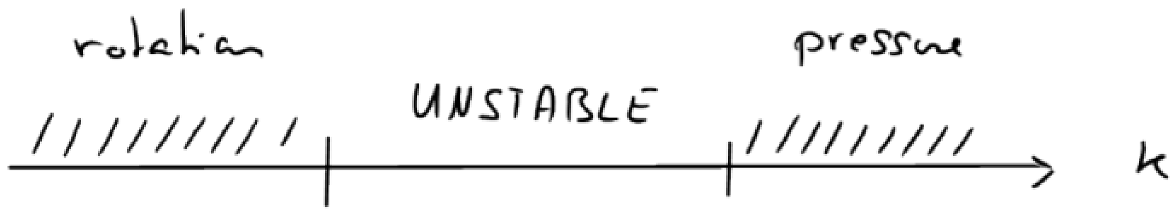
$|k| < k_{crit}$ ($\lambda > \lambda_{crit}$) $\omega^2 < 0$ **UNSTABLE**

The pressure stabilizes the system at small scale

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

Rotating fluid disk

$$\omega^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$



$$k_{c,1} = \frac{\kappa^2}{2\pi G \Sigma_0} \qquad k_{c,2} = \frac{2\pi G \Sigma_0}{v_s^2}$$

Complete stability

$$k_1 \gg k_2$$

$$Q := \frac{\kappa v_s}{\pi G \Sigma_0} > 1$$

Stability of stellar disks (using a Schwarzschild DF)

$$Q := \frac{\kappa v_s}{\pi G \Sigma_0} > 1$$

$$Q := \frac{\kappa \sigma_R}{3.36\, G \Sigma_0} > 1$$

Safronov - Toomre criterion (1964)

Non axisymmetric perturbations

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

$$\omega = \sqrt{\kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2} + m\Omega$$

- the stability is determined for $m = 0$ (value of $\sqrt{\ }$)
- $m\Omega \in \mathbb{R}e$ → add an oscillatory term $e^{-i m\Omega}$ with a frequency that correspond to the passage of spiral arm

Non axisymmetric perturbations

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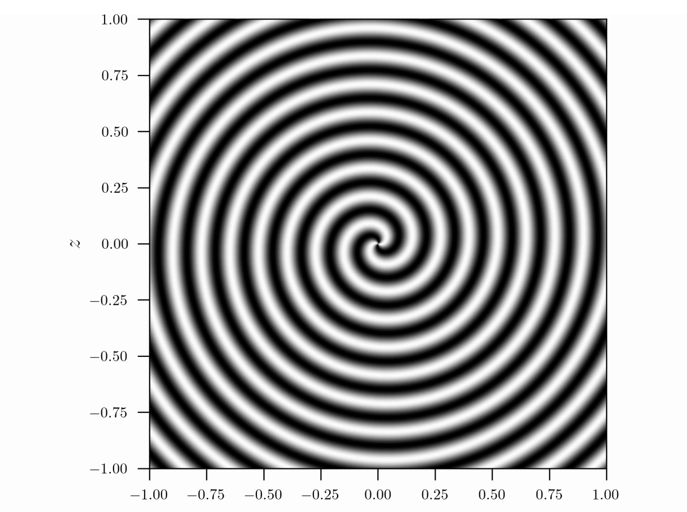
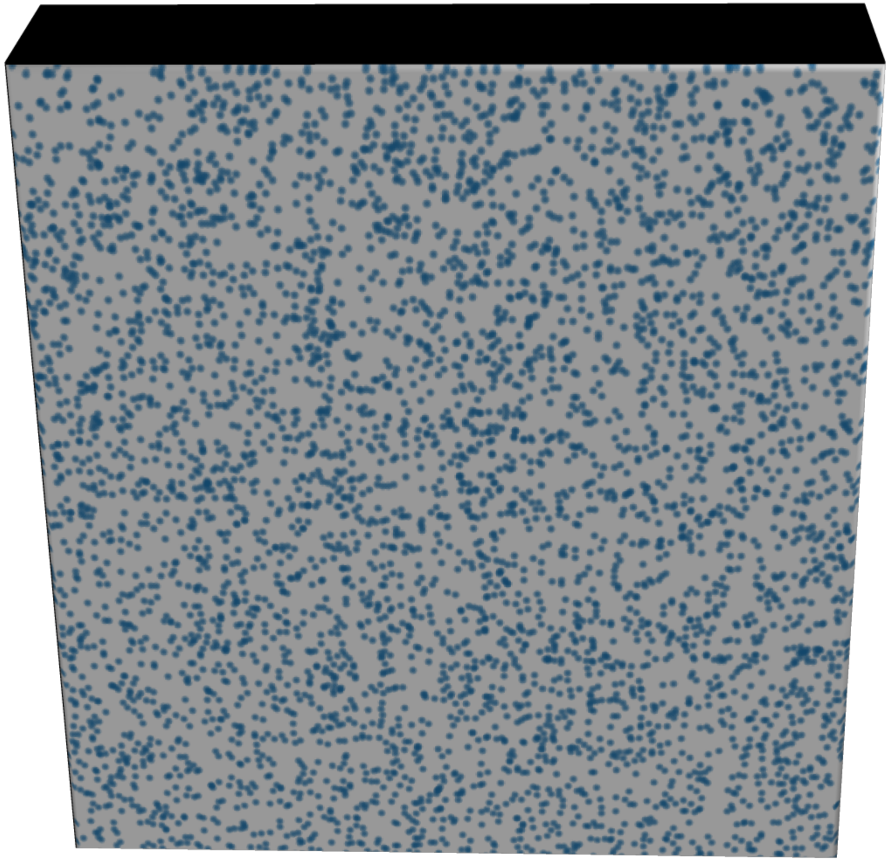


The origin of spiral structures :

another view

(Jeans 1902)



- Infinite slab of infinite thickness, homogeneous Σ
- Particles with random velocities (constants vel. dispersion σ)
- Gravitational interactions only (collisonless system).

The Jeans instability

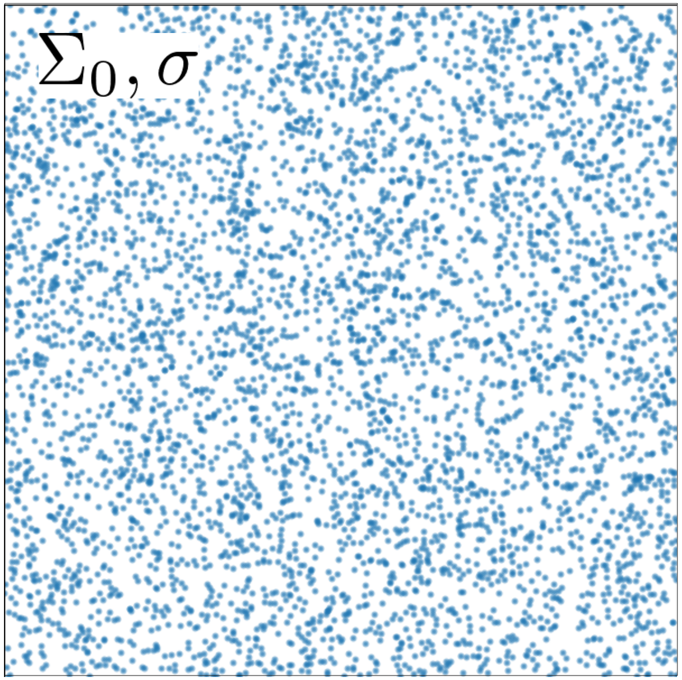
(Jeans 1902)

- Infinite razor thin medium

$$\lambda_\mathrm{J} = 2\, r_\mathrm{J} = \frac{\sigma^2}{G\Sigma_0}$$

$\lambda > \lambda_J$ $\quad\quad$ $\lambda < \lambda_J$

unstable $\quad\quad$ stable



The Jeans instability

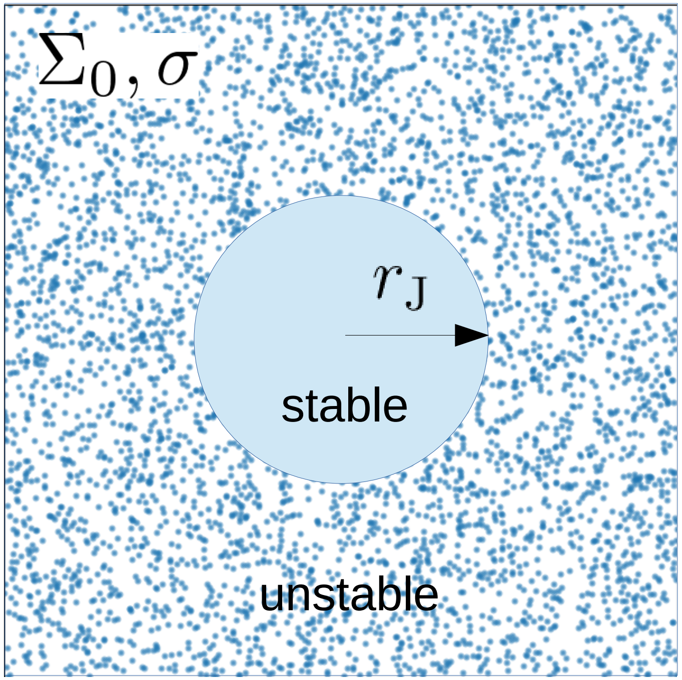
(Jeans 1902)

- Infinite razor thin medium

$$\lambda_J = 2\,r_J = \frac{\sigma^2}{G\Sigma_0}$$

$r > r_J$ — <u>unstable</u>

$r < r_J$ — <u>stable</u>



The link with the virial equilibrium

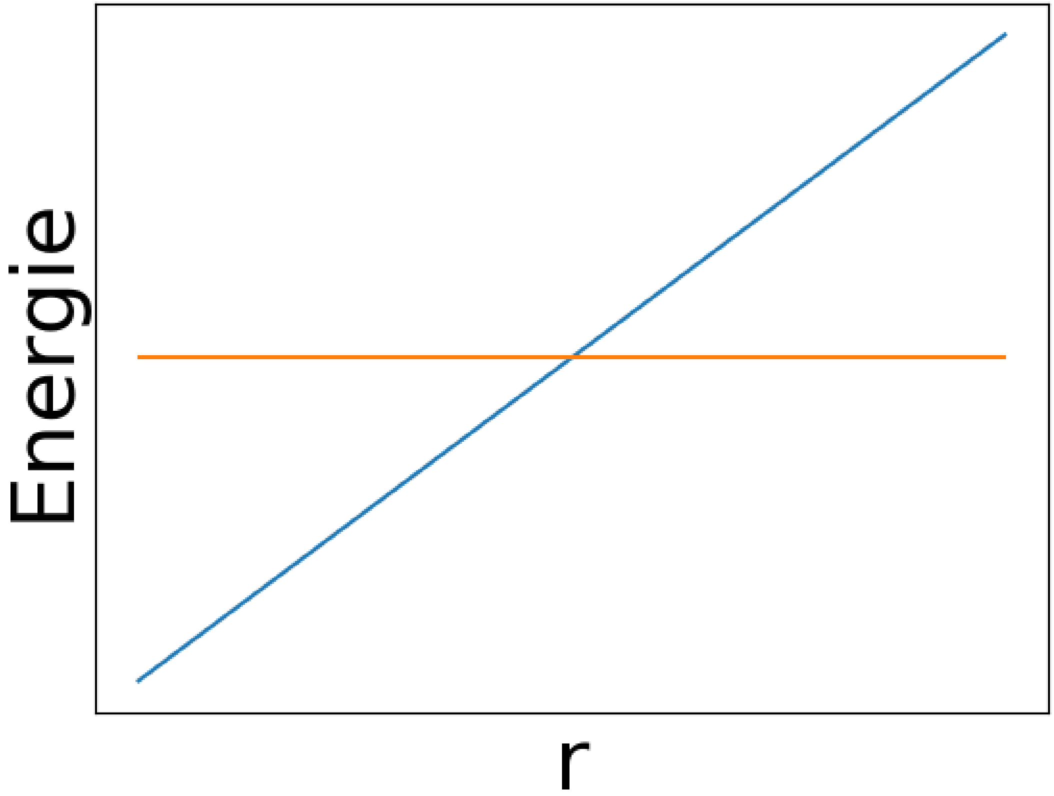
$$\sigma^2 = \frac{2}{\pi}\frac{GM_J}{r_J}$$

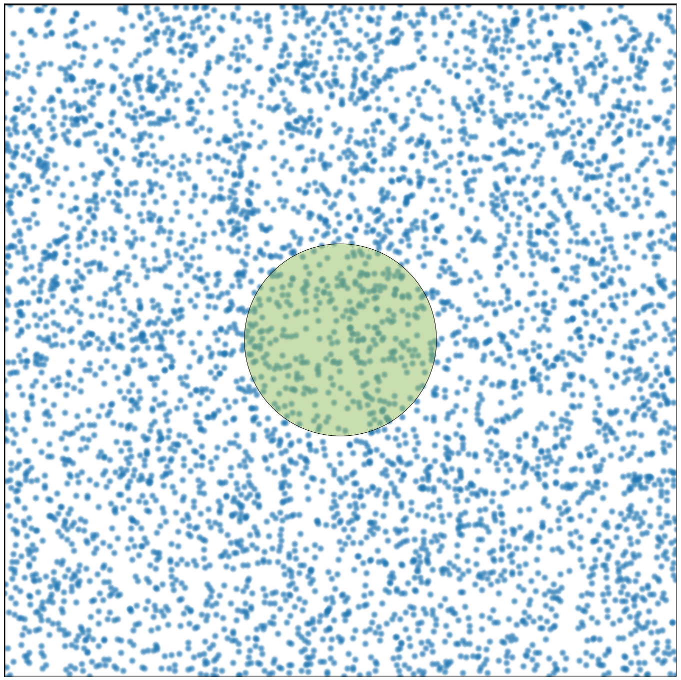
with

$$M_J = \pi r_J^2 \Sigma_0$$

The Jeans instability

(Jeans 1902)

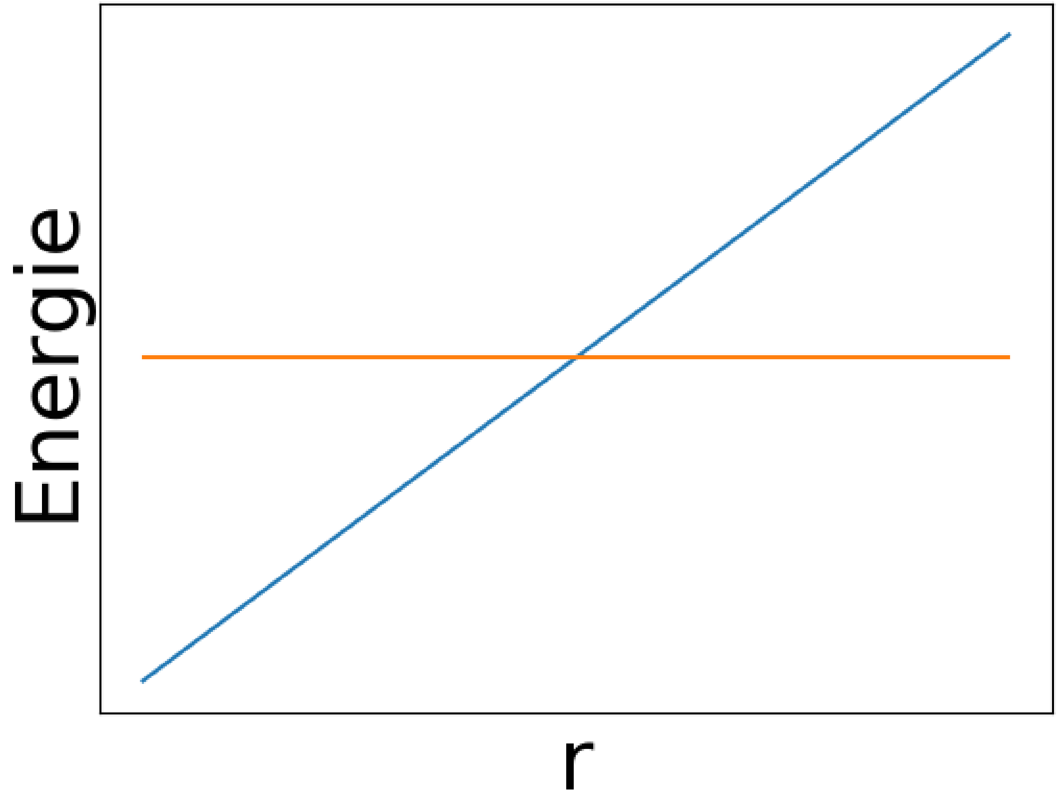


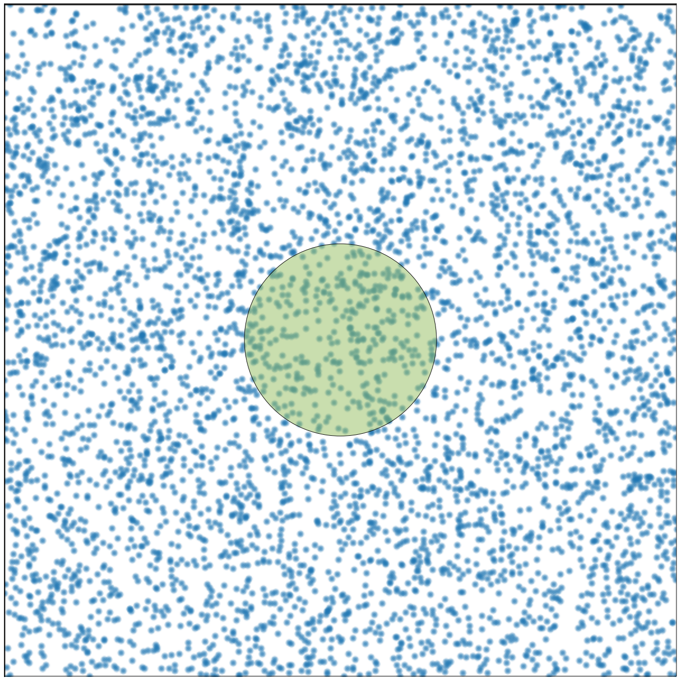


$$K \sim \sigma^2$$

The Jeans instability

(Jeans 1902)

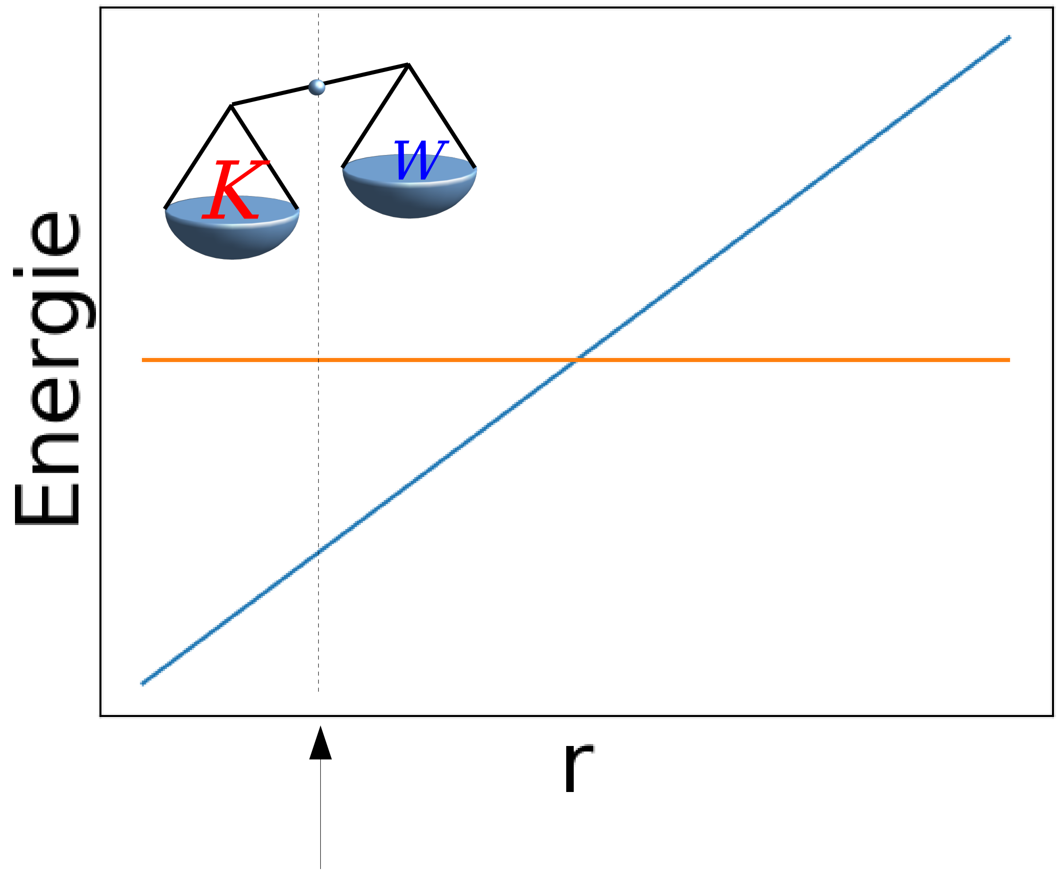


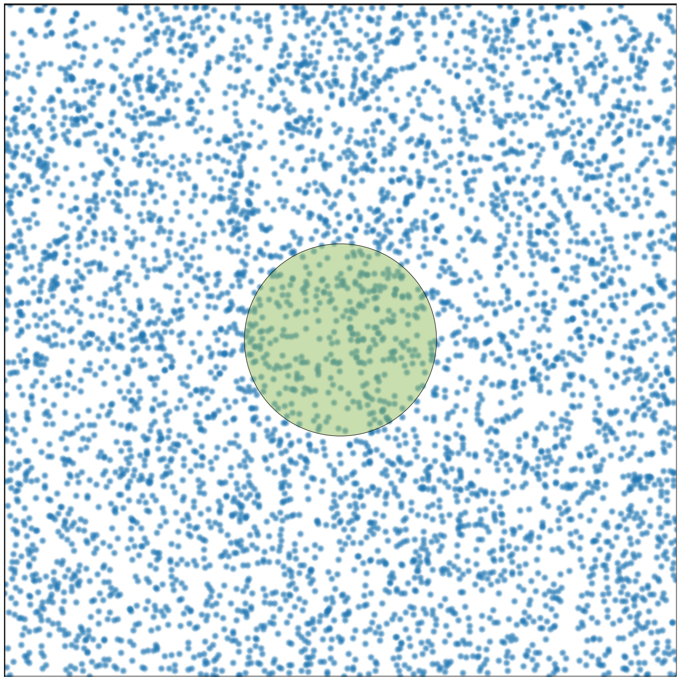


$$K \sim \sigma^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

The Jeans instability

(Jeans 1902)





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The Jeans instability

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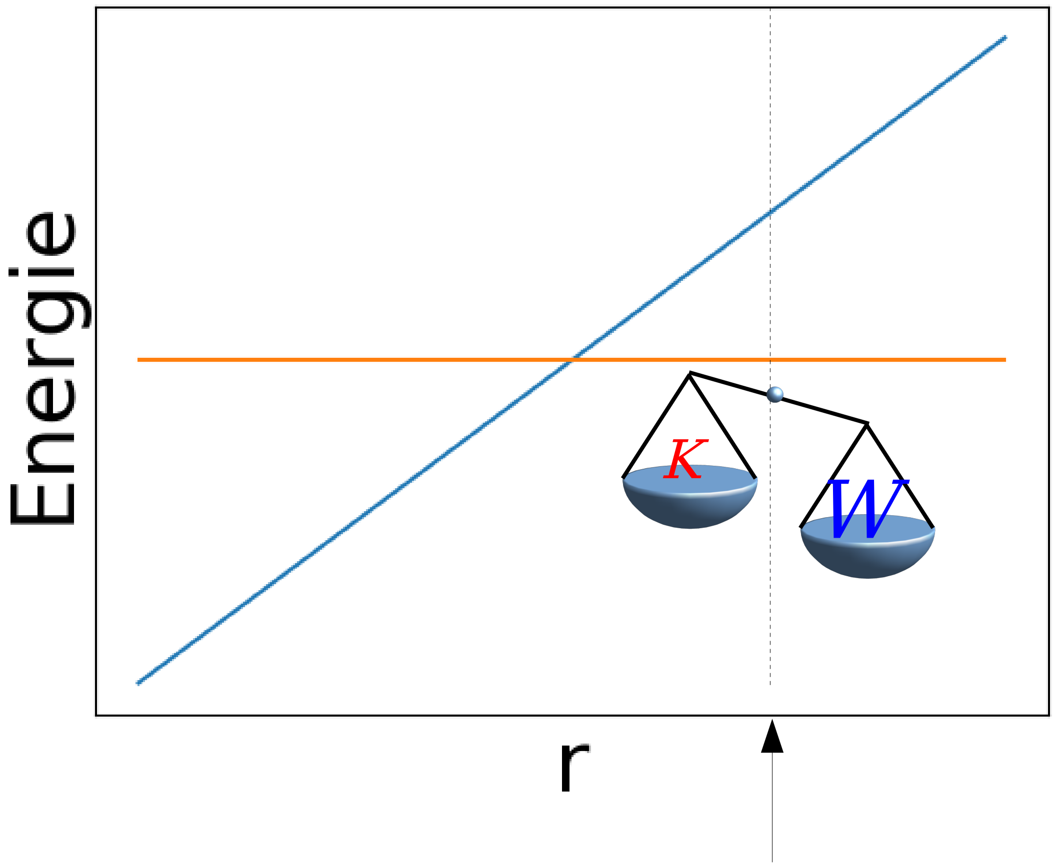
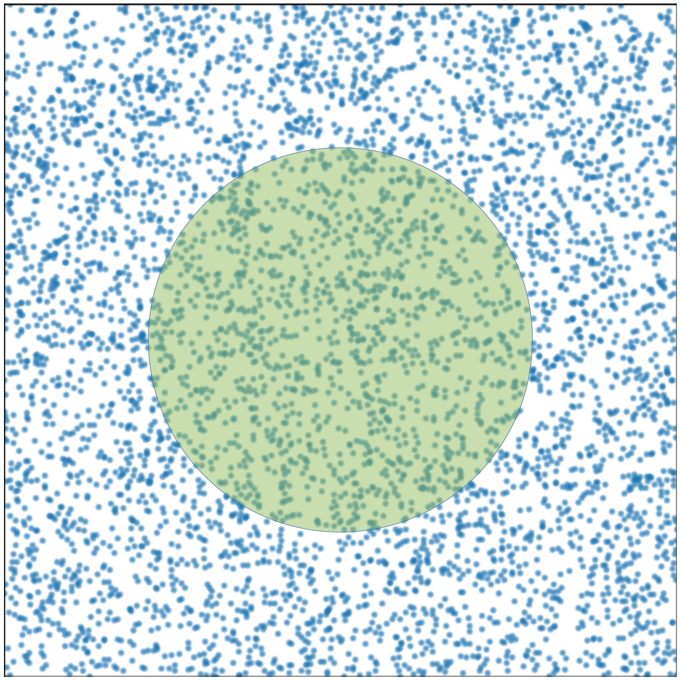




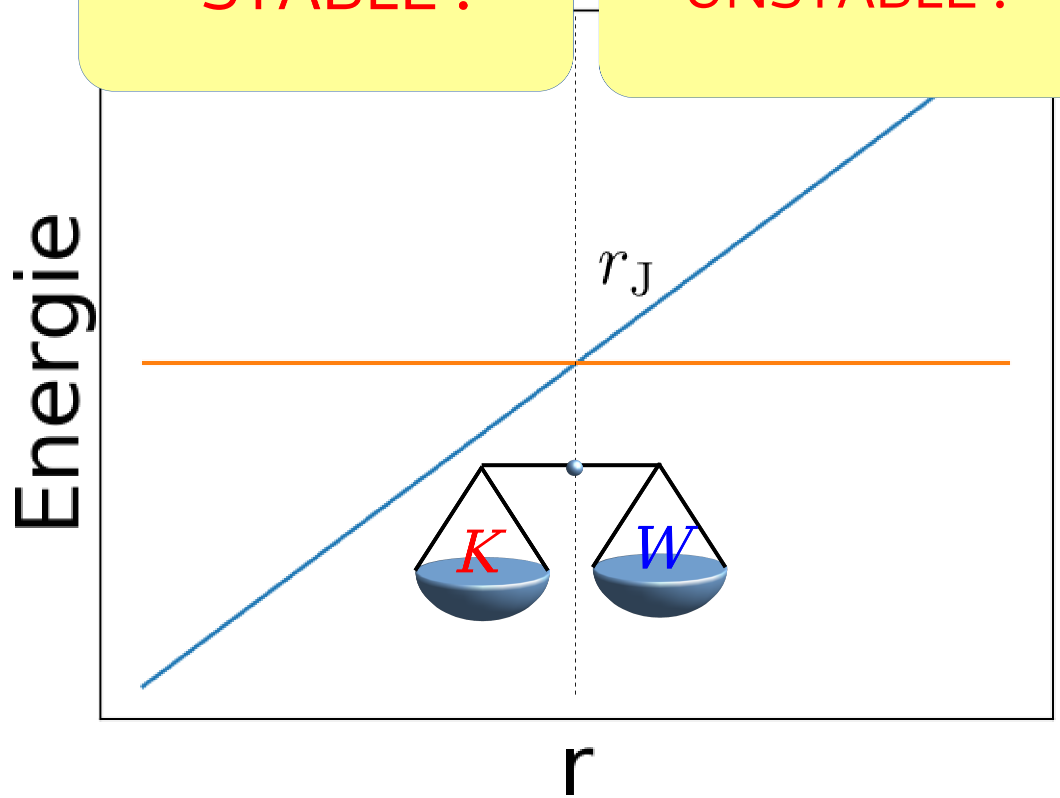
$$K \sim \sigma^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

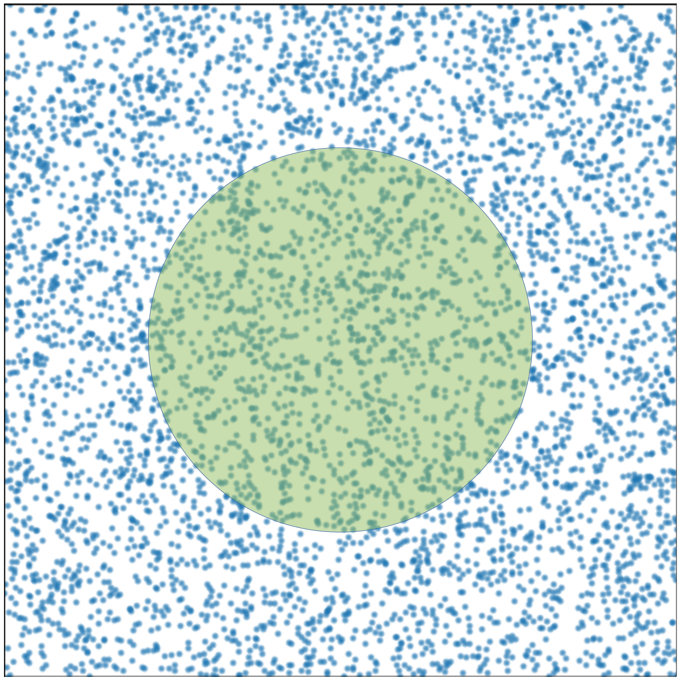
The Jeans instability

(Jeans 1902)

STABLE !

UNSTABLE !

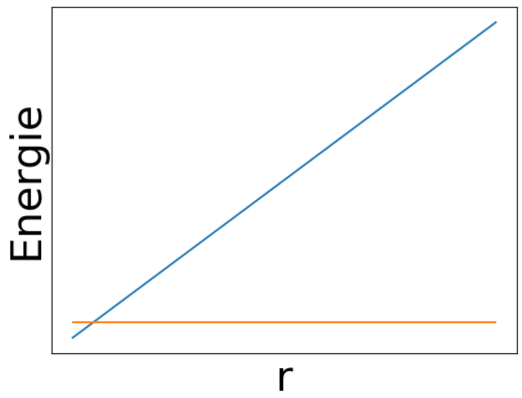




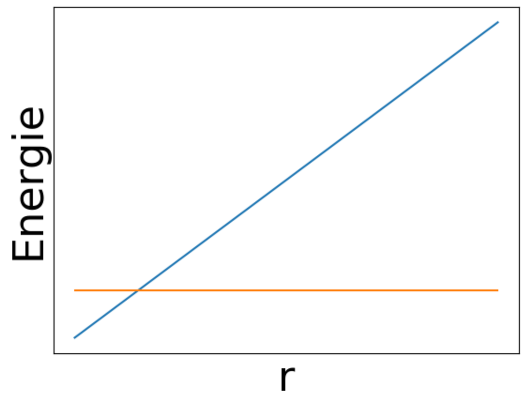
$$K \sim \sigma^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

The Jeans instability in an infinite razor-thin sheet

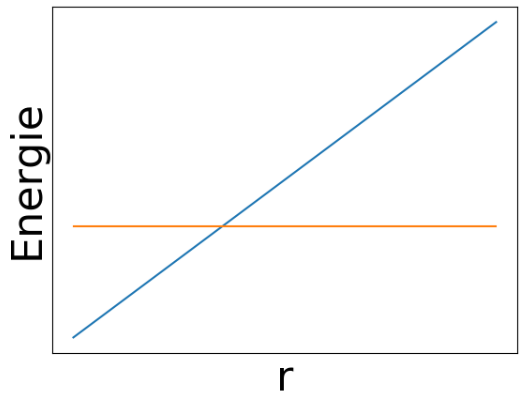
$\sigma = 0.1, r_{\mathrm{J}} = 0.005$

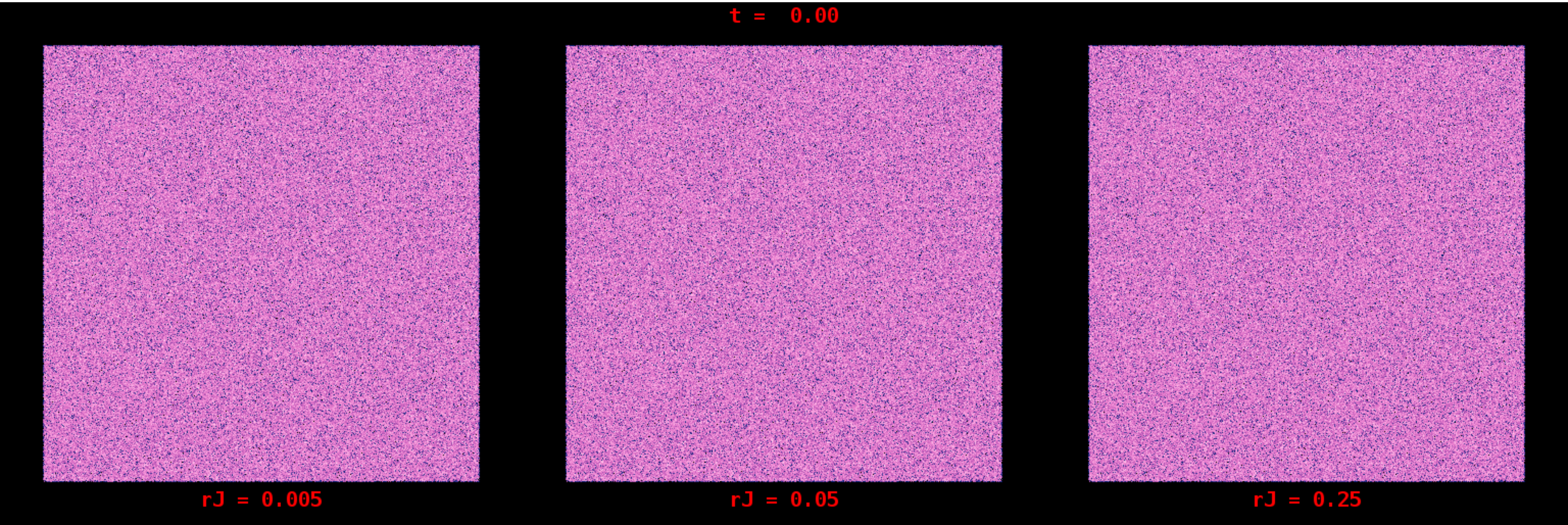


$\sigma = 0.3, r_{\mathrm{J}} = 0.05$



$\sigma = 0.7, r_{\mathrm{J}} = 0.25$





Can we stabilize a razor-thin sheet against gravitational instabilities using non-random motions ?

$$2T = K$$

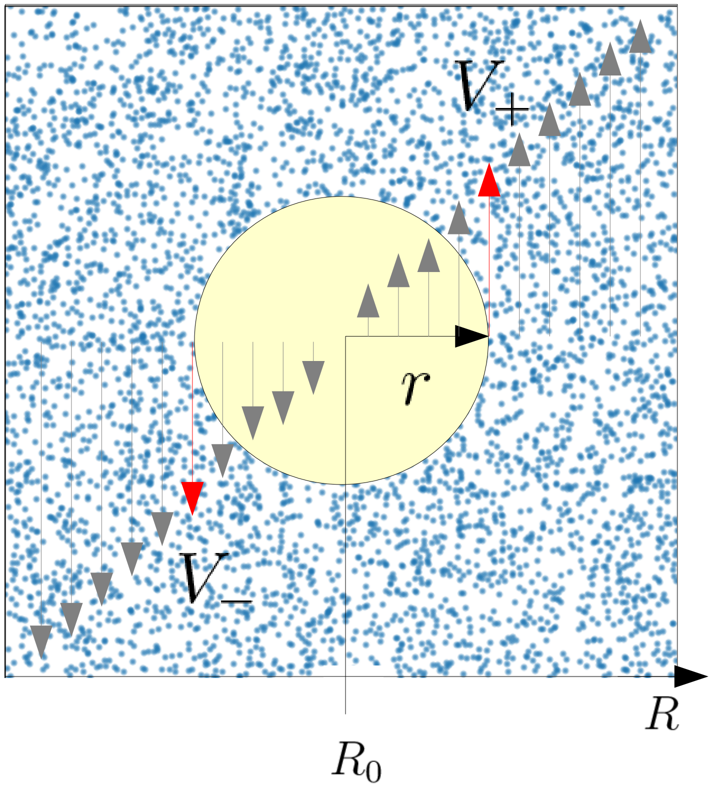
Stabilizing using shearing motions

$V(R)$: a vertical velocity field (aligned with the y axis)

At the edges of a disk

$$V_+ = V(R_0) + \left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right) r$$

$$V_- = V(R_0) - \left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right) r$$



- Kinetic energy

$$2\,E_{\mathrm{kin}} \cong \Delta V^2 = (V_+ - V_-)^2 = 4\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right)^2 r^2$$

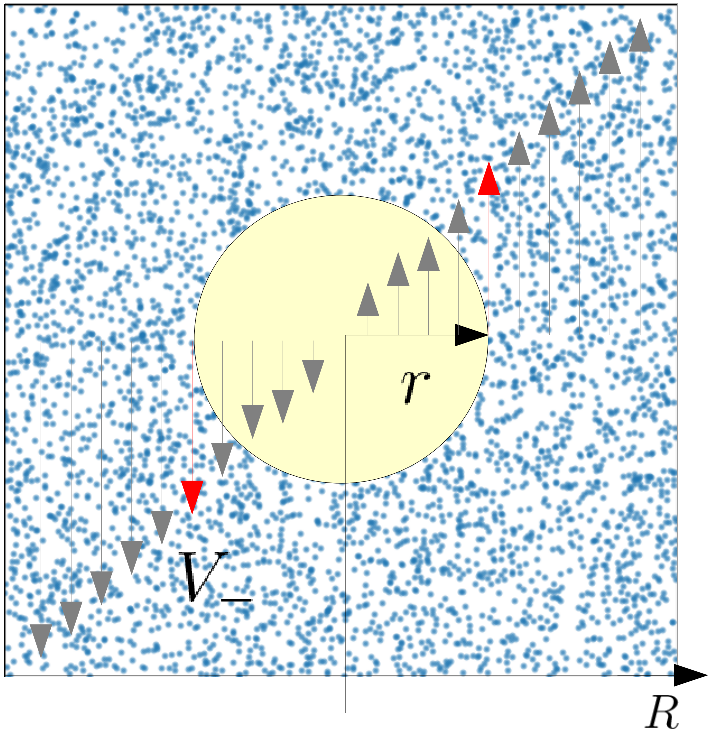
Stabilizing using shearing motions

- Kinetic energy

$$2\,E_{\rm kin} = \frac{1}{\pi\,r^2}\int_S V(R)^2 ds$$

$$V(R)^2 \cong V^2(R_0) + \left(\left.\frac{\rm dV}{\rm dR}\right|_{R_0}\right)^2 (R-R_0)^2 + 2\,V(R_0)\left.\frac{\rm dV}{\rm dR}\right|_{R_0}(R-R_0)$$

$$2\,E_{\rm kin} = V^2(R_0) + \frac{1}{2}\left(\left.\frac{\rm dV}{\rm dR}\right|_{R_0}\right)^2 r^2$$



Stabilizing using shearing motions

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$$2\,E_{\rm kin} = \frac{1}{\pi\, r^2}\int_S V(R)^2 ds$$

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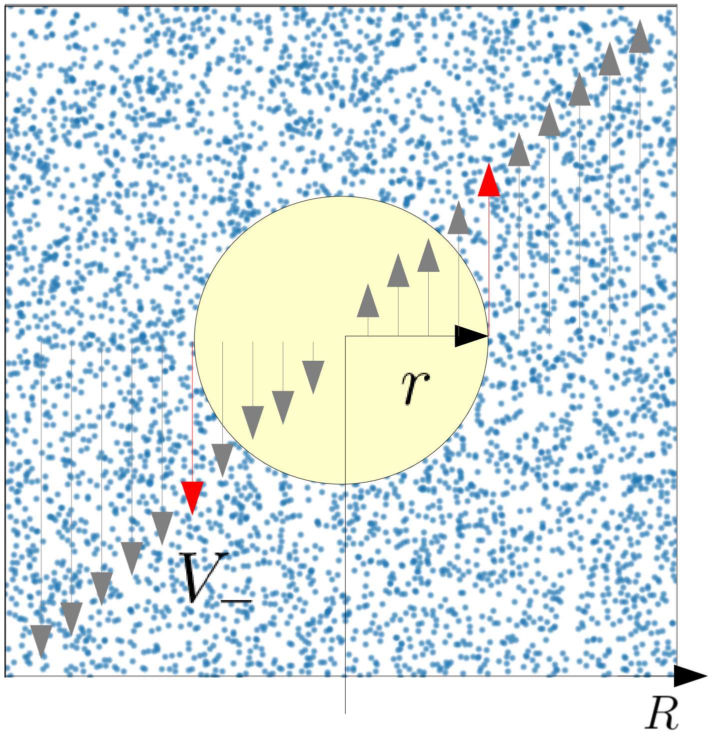
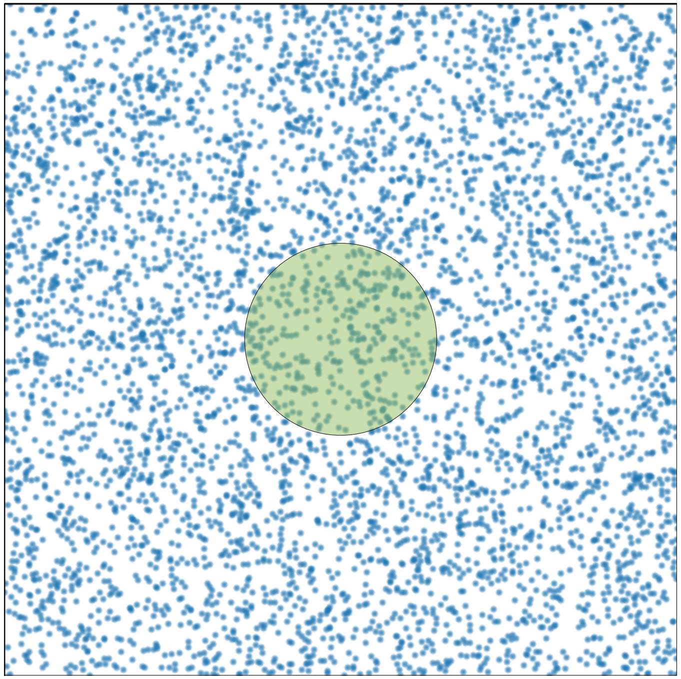
$$2\,E_{\rm kin} = V^2(R_0) + \frac{1}{2}\left(\frac{\rm dV}{\rm dR}\bigg|_{R_0}\right)^2 r^2$$

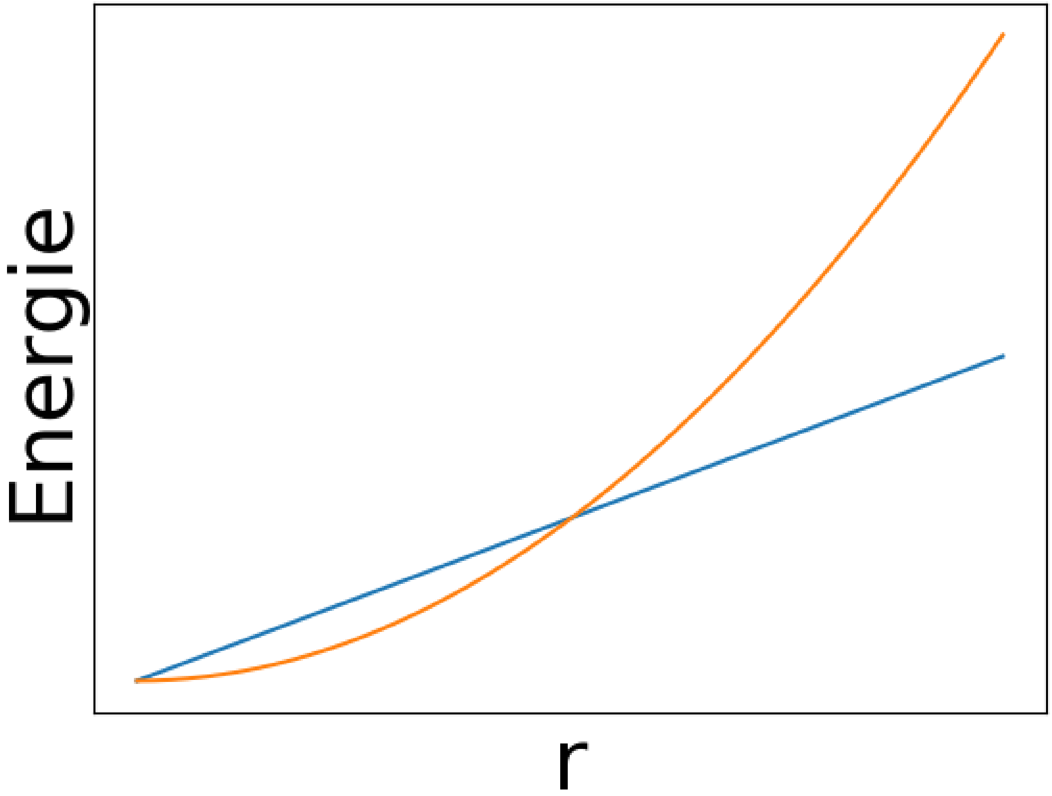


- Potential energy

$$E_{\rm pot} = \frac{G\,M_S}{r} = \frac{G\,\Sigma\,\pi\,r^2}{r}$$

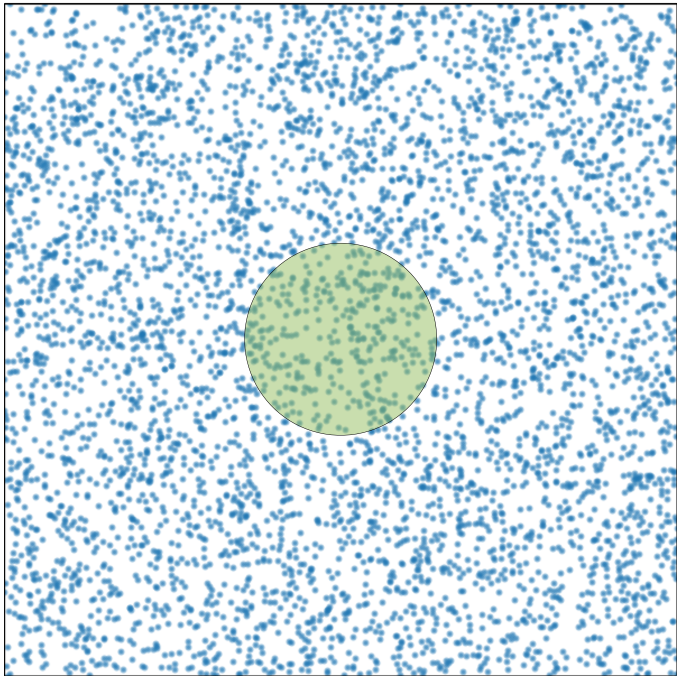
Stabilizing using shearing motions

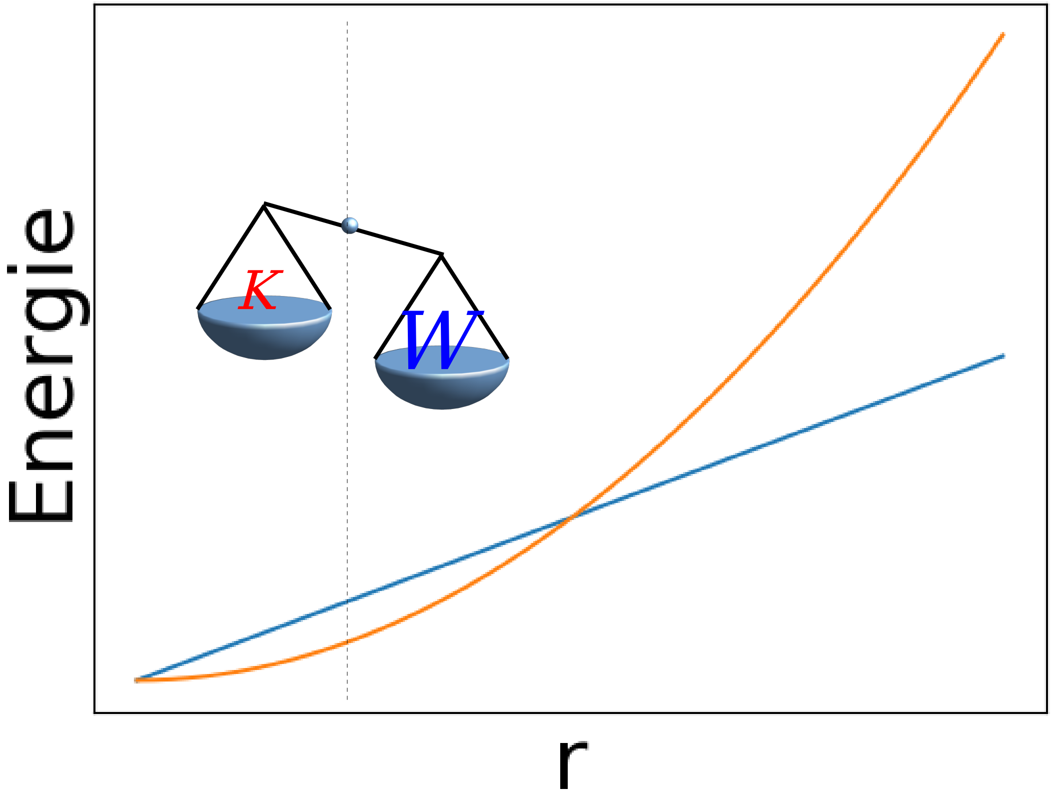




$$K \sim \left(\frac{\mathrm{dV}}{\mathrm{dR}} \Big|_{R_0} \right)^2 r^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

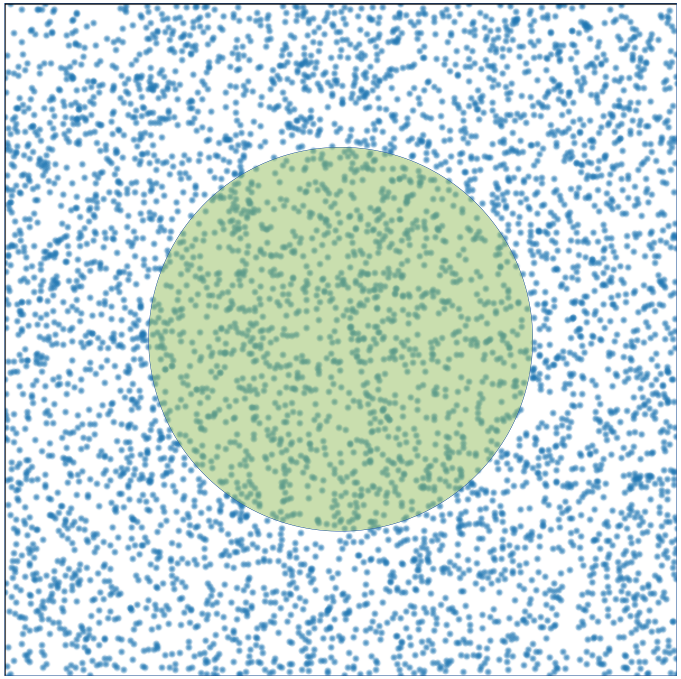
Stabilizing using shearing motions

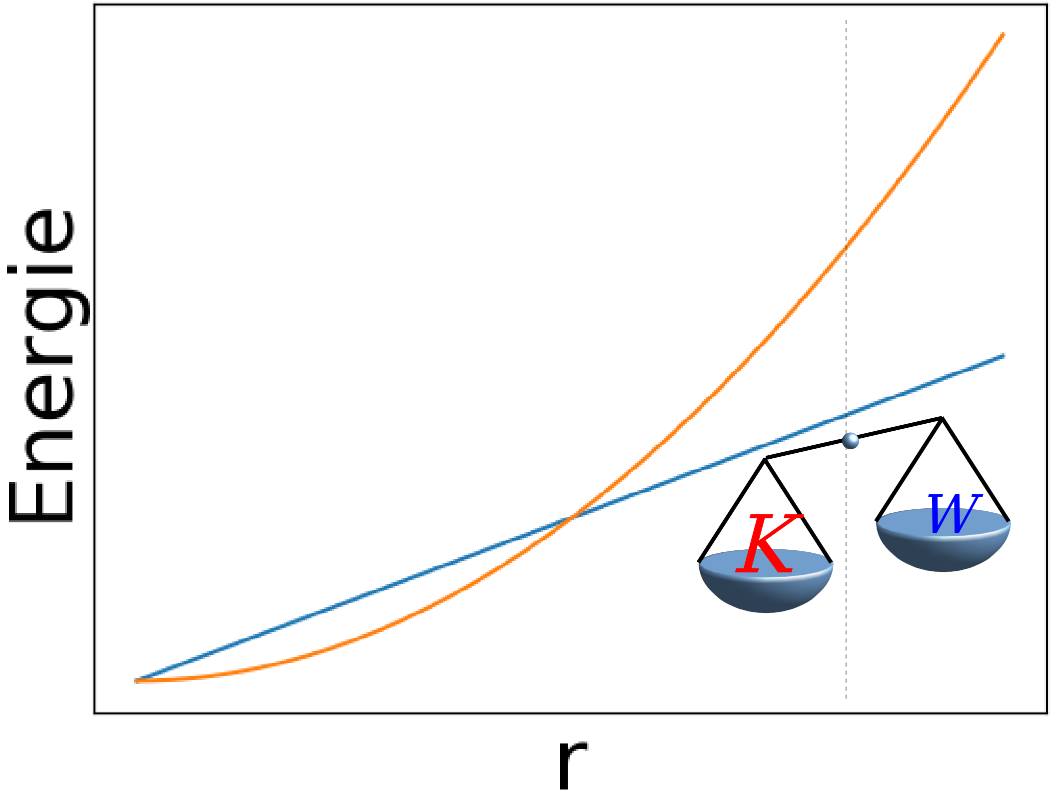




$$K \sim \left(\frac{\mathrm{dV}}{\mathrm{dR}} \bigg|_{R_0} \right)^2 r^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

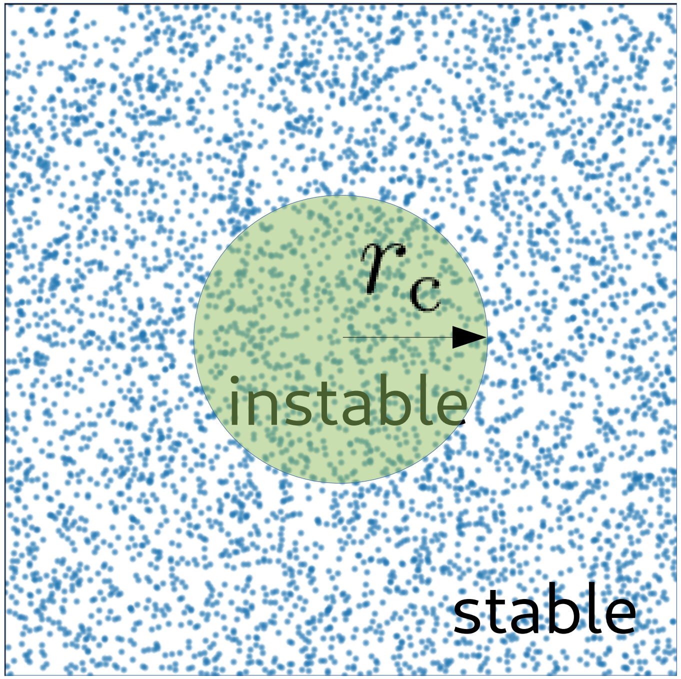
Stabilizing using shearing motions

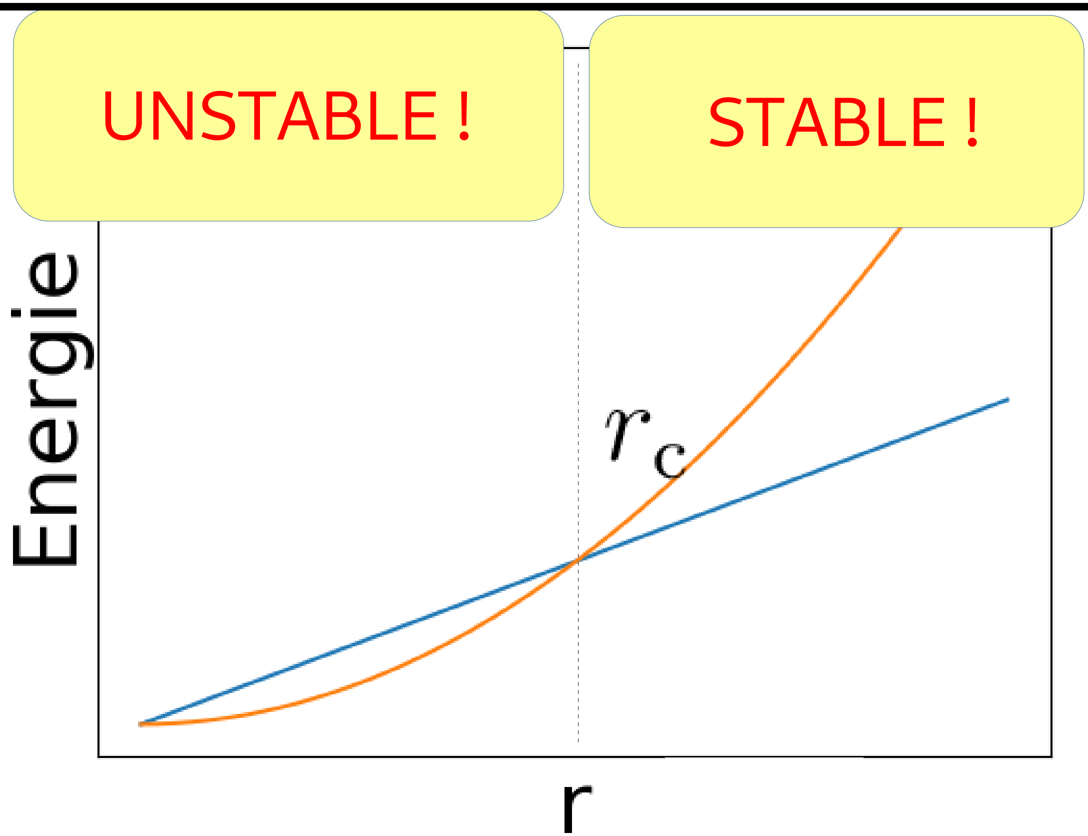




$$K \sim \left(\left. \frac{\mathrm{dV}}{\mathrm{dR}} \right|_{R_0} \right)^2 r^2 \qquad W \sim \frac{G\,M(r)}{r} = G\Sigma\pi r$$

Stabilizing using shearing motions





$$\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right)^2 = 2\pi\,\frac{G\,\Sigma}{r}$$

$$r_c = 2\pi\,\frac{G\,\Sigma}{\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right)^2}$$

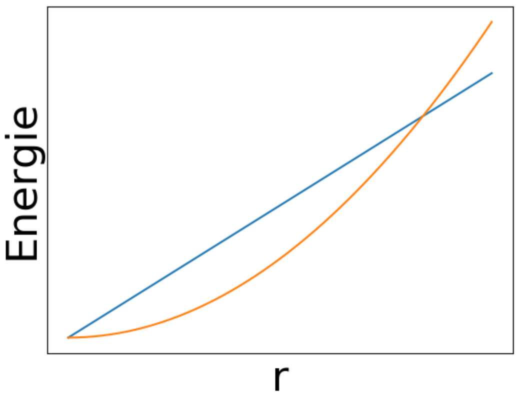
$r > r_c$ stable

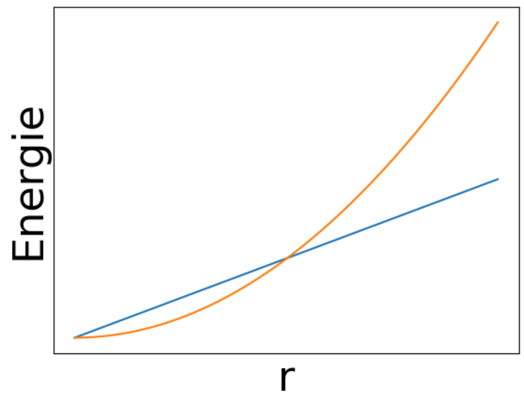
$r < r_c$ unstable

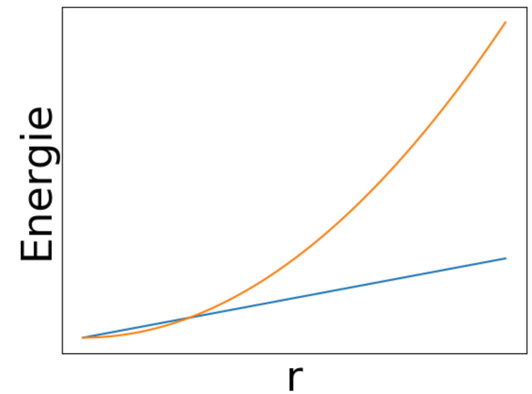
- r_c critical radius
- Beyond this radius, its no longer possible for a perturbation to growth

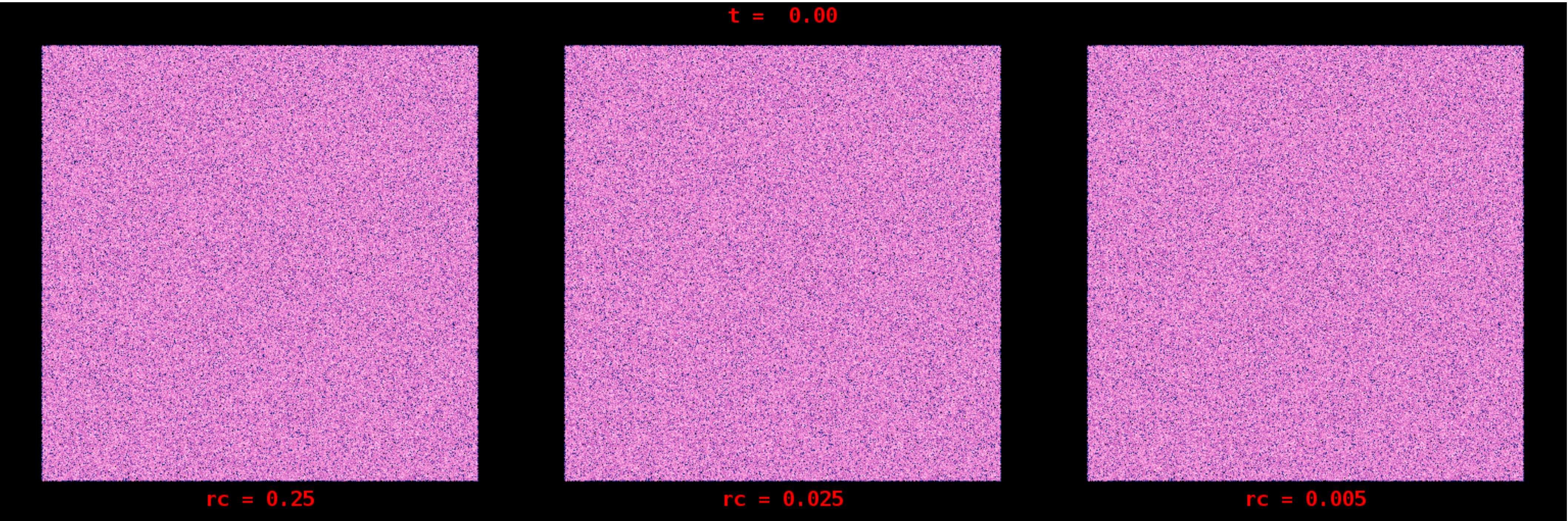
$V_{\text{vert.}}(R) = \alpha R$ $\sigma = 0$

$r_c = 0.25$ $r_c = 0.025$ $r_c = 0.005$





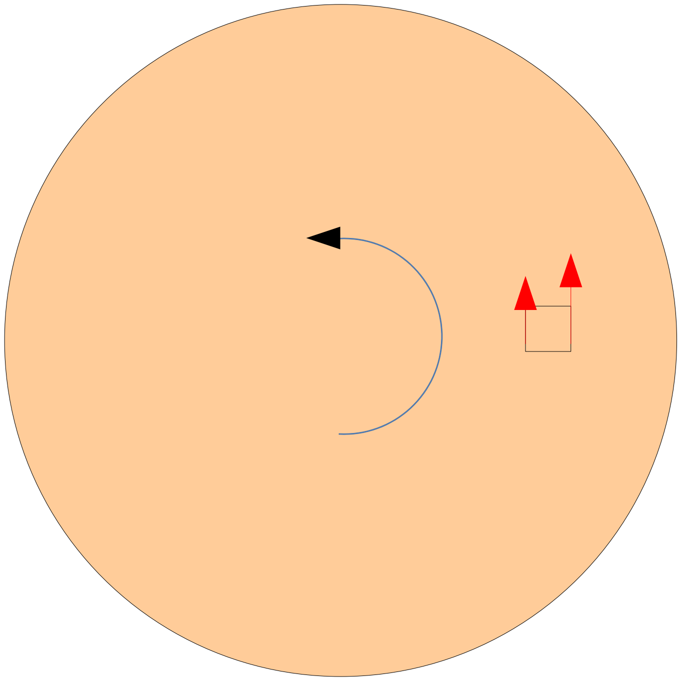




What is the link between slabs and rotating disks ?

Rotating disks of infinite thickness : I - rigid rotation

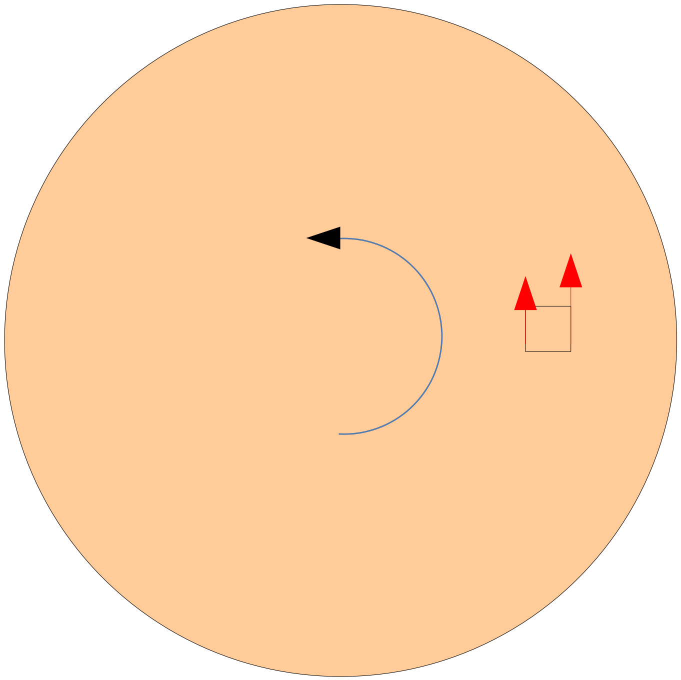
$$\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right)^2 = 2\pi\frac{G\Sigma}{r} \qquad \left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0} = \Omega_0$$



$$V = \Omega \cdot r$$

Rotating disks of infinite thickness : I - rigid rotation

$$\left(\left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0}\right)^2 = 2\pi\,\frac{G\,\Sigma}{r} \qquad \left.\frac{\mathrm{dV}}{\mathrm{dR}}\right|_{R_0} = \Omega_0$$



$$\Omega_0^2 = 2\pi\,\frac{G\,\Sigma}{r}$$

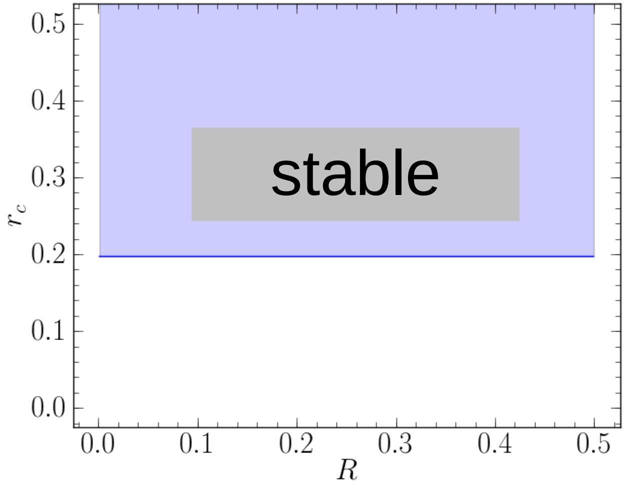
$$\frac{1}{2}\Omega_0^2\,r^2 = \frac{G\,M}{r}$$

Critical radius

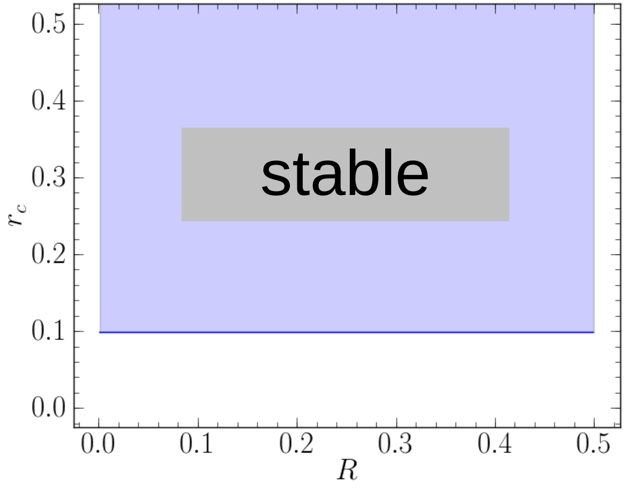
$$r_c = 2\pi\,\frac{G\,\Sigma}{\Omega_0^2}$$

Rotating disks of infinite thickness : I - rigid rotation

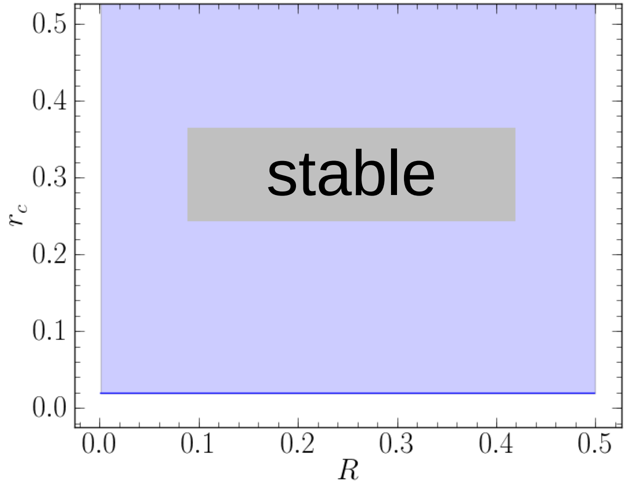
$r_c = 0.25, \sigma = 0$

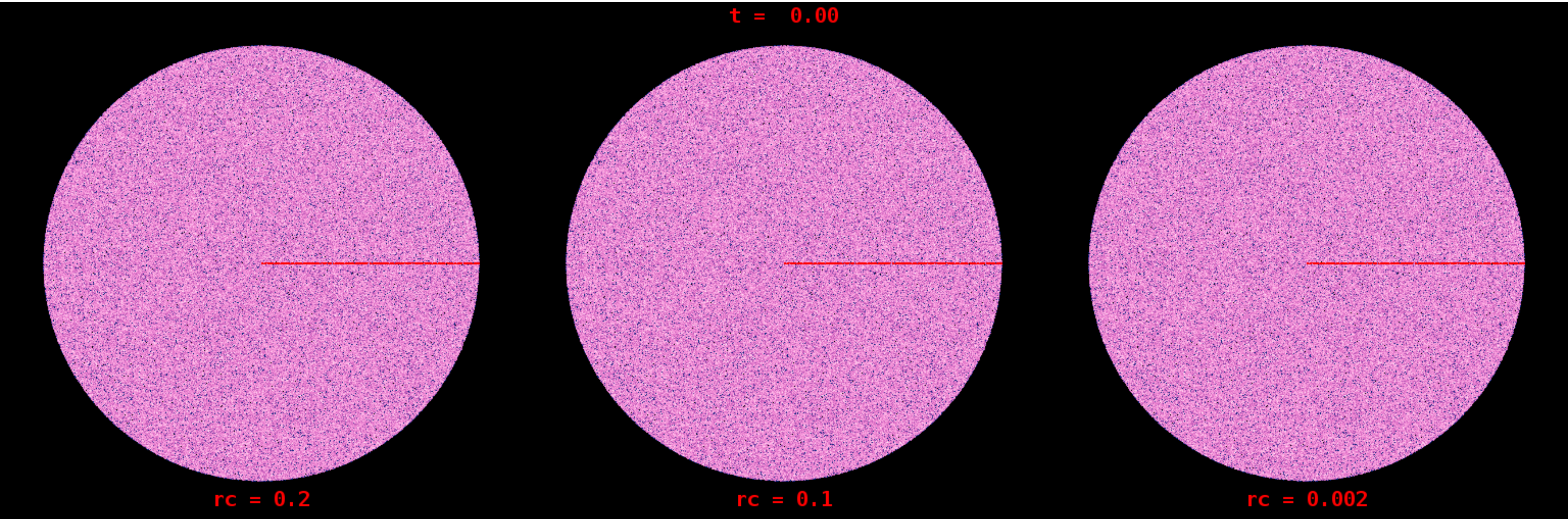


$r_c = 0.025, \sigma = 0$



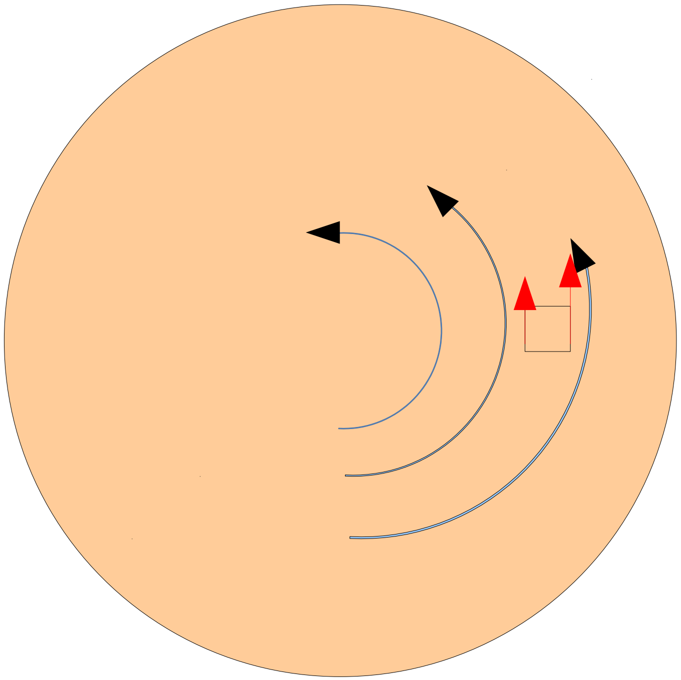
$r_c = 0.005, \sigma = 0$





Rotating disks of infinite thickness : II - differential rotation

Need to develop the velocity up to the second order



Rotating disks of infinite thickness : II - differential rotation

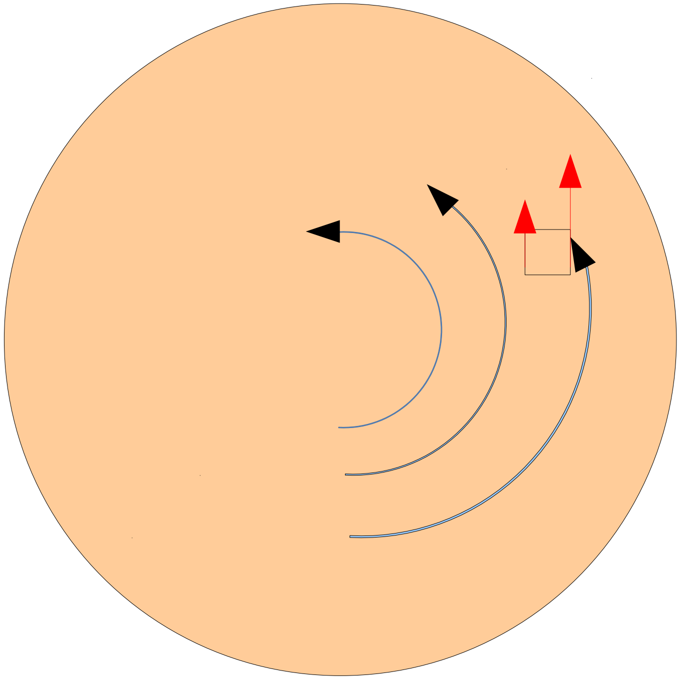
Need to develop the velocity up to the second order

$$\kappa^2 = 8\pi \frac{G\,\Sigma}{r}$$

$$\frac{1}{8}\kappa^2\, r^2 = \frac{G\,M}{r}$$

Critical radius

$$r_c = 8\pi \frac{G\,\Sigma}{\kappa^2}$$



Rotating disks of infinite thickness : II - differential rotation

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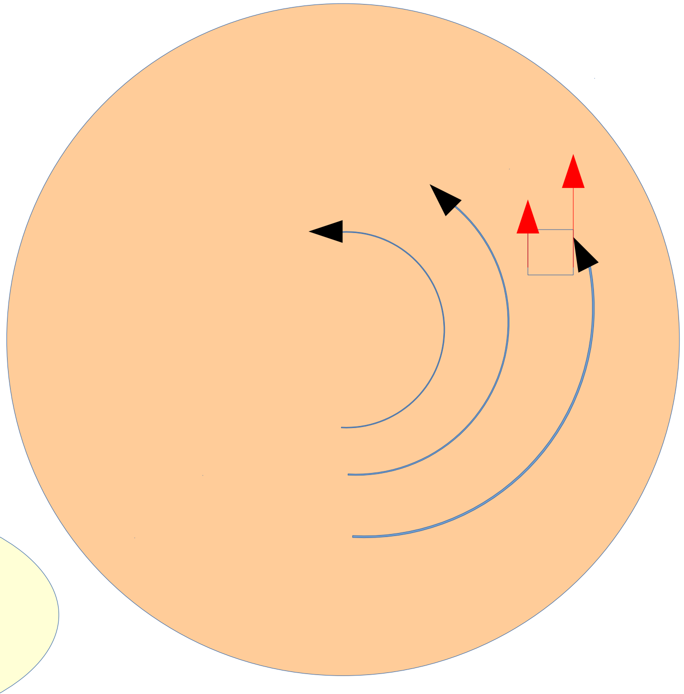
$$\kappa^2 = 8\pi \frac{G\,\Sigma}{r}$$

$$\frac{1}{8}\kappa^2\, r^2 = \frac{G\,M}{r}$$



Critical radius

$$r_c = 8\pi \frac{G\,\Sigma}{\kappa^2}$$

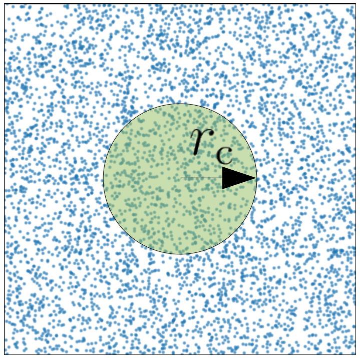
Classical derivation

$$\omega^2 = \kappa^2 - 2\pi\, G\,\Sigma\,|k|$$

$$r_c = \frac{\lambda_c}{2} = 2\pi^2 \frac{G\,\Sigma}{\kappa^2}$$

Predicting the number of spiral arms...

Rotating disks of infinite thickness : the number of spiral arms



- For a rotating disk of a given surf. density Σ

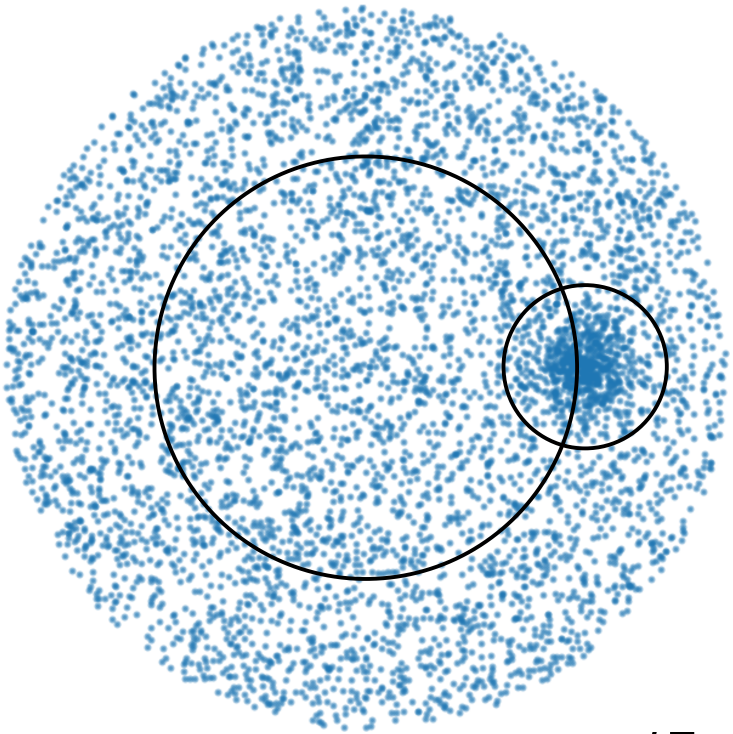
  the radial epicycle frequency κ

  determines the maximal size

  of the clumps

$$r_c = 8\pi \frac{G\,\Sigma}{\kappa^2}$$

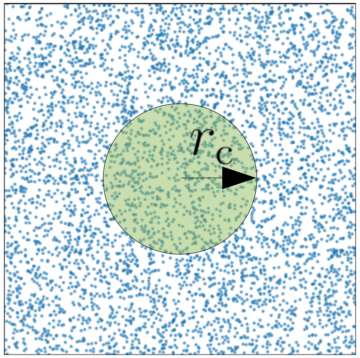
- Number of clumps per radius



$$N_{\rm c} = \frac{L}{2\,r_c} = \frac{\kappa^2\,R}{8\,G\,\Sigma}$$

$$L = 2\pi\,R$$

Rotating disks of infinite thickness : the number of spiral arms



- For a rotating disk of a given surf. density Σ

  the radial epicycle frequency κ

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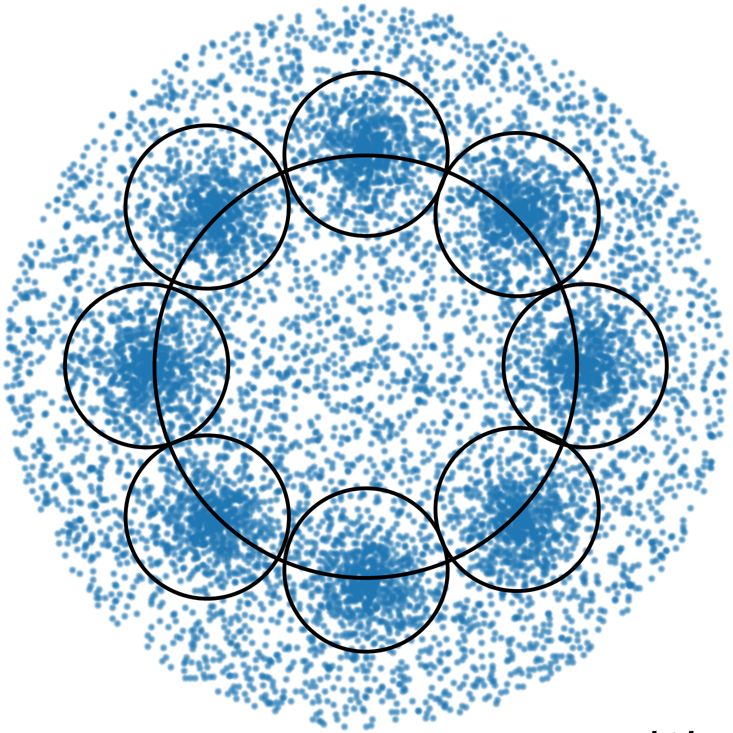
$$r_c = 8\pi \frac{G\Sigma}{\kappa^2}$$

- Number of clumps per radius

$$N_c = \frac{L}{2\,r_c} = \frac{\kappa^2\,R}{8\,G\,\Sigma}$$

$$L = 2\pi\,R$$



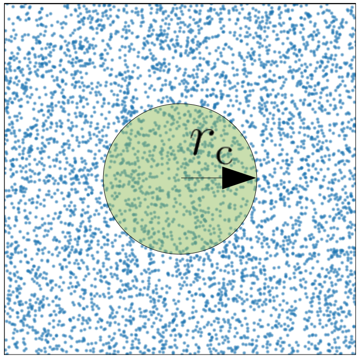
Rotating disks of infinite thickness : the number of spiral arms

- For a rotating disk of a given surf. density Σ

  the radial epicycle frequency κ

  determines the maximal size

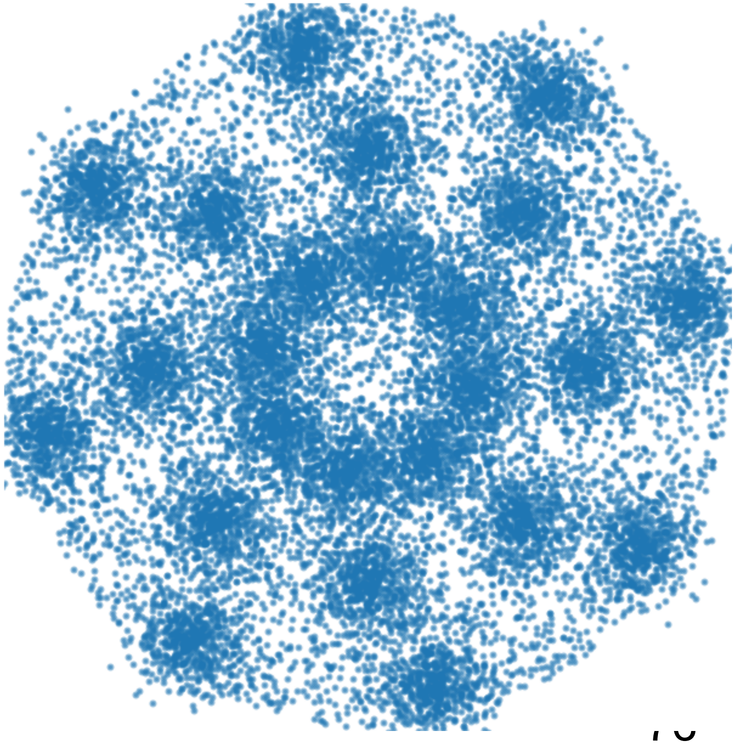
  of the clumps $r_c = 8\pi \dfrac{G\,\Sigma}{\kappa^2}$



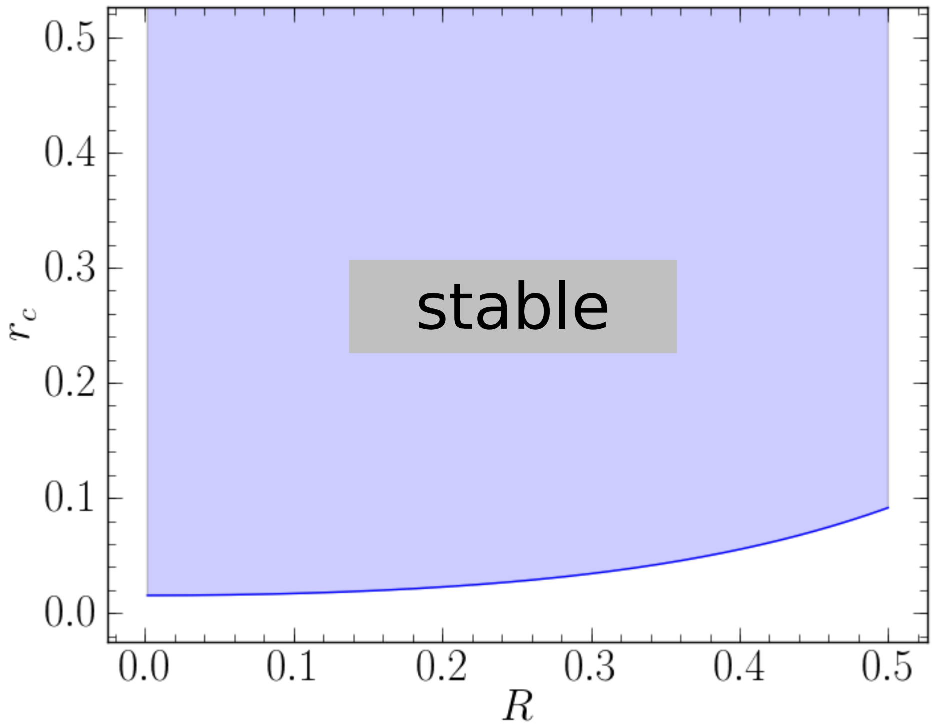
- Number of clumps per radius

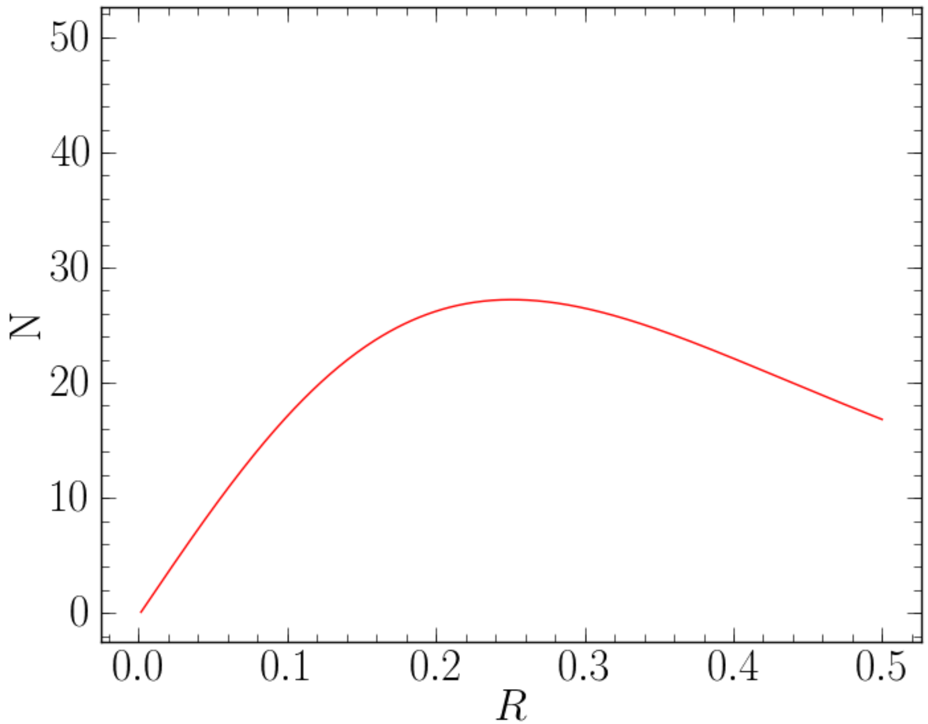
$$N_c = \frac{L}{2\,r_c} = \frac{\kappa^2\,R}{8\,G\,\Sigma}$$

$$L = 2\pi\,R$$



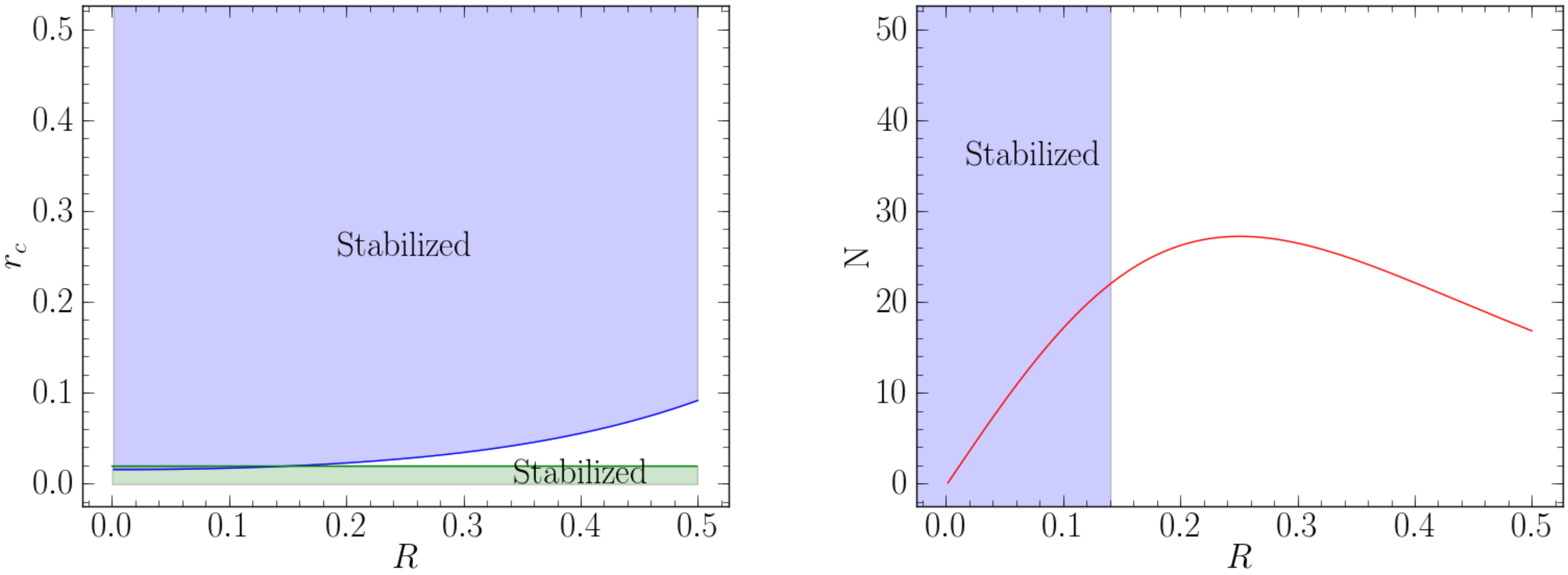
Rotating disks of infinite thickness : the number of spiral arms



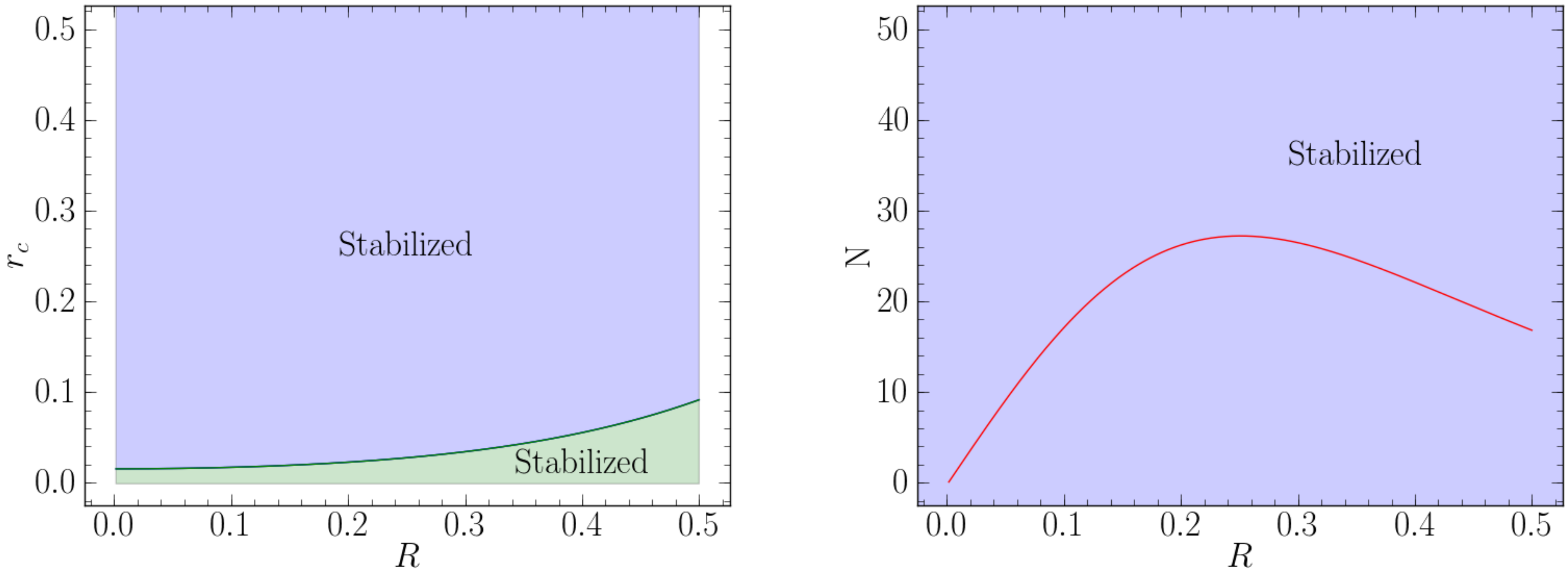


Putting all together...

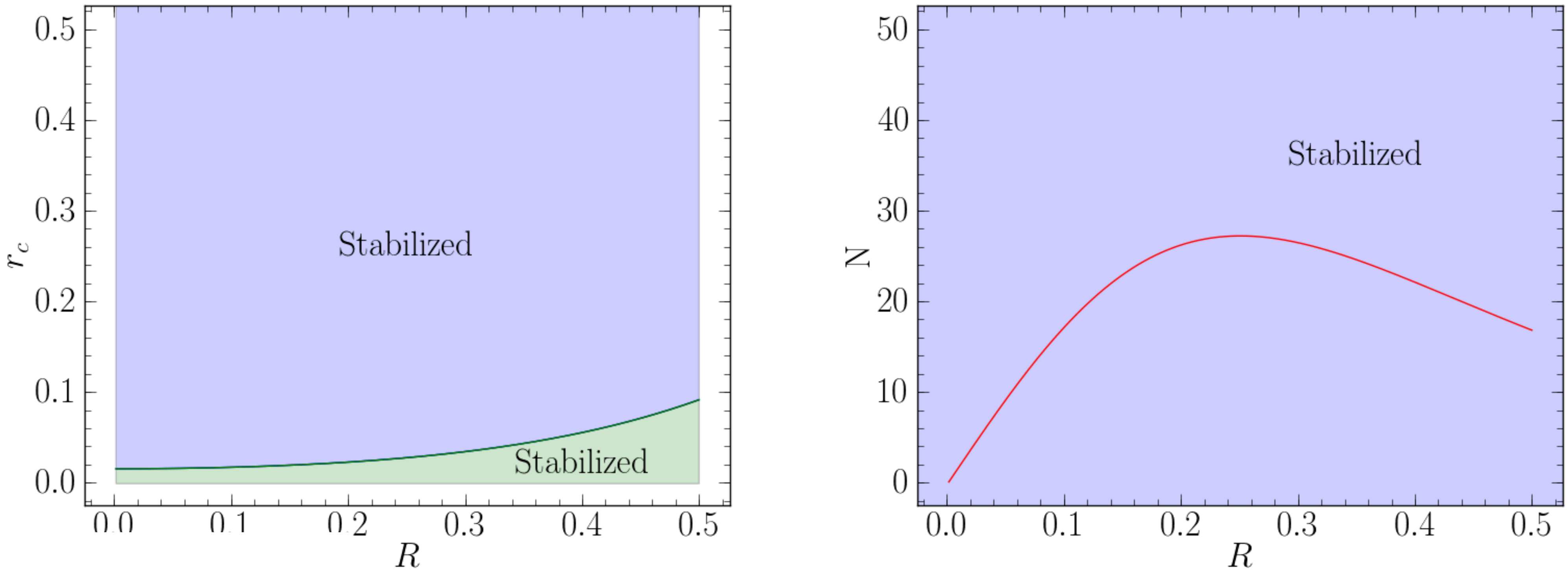
Rotating disks of infinite thickness : adding velocity dispersion



Rotating disks of infinite thickness : Local stability



Rotating disks of infinite thickness : Local stability



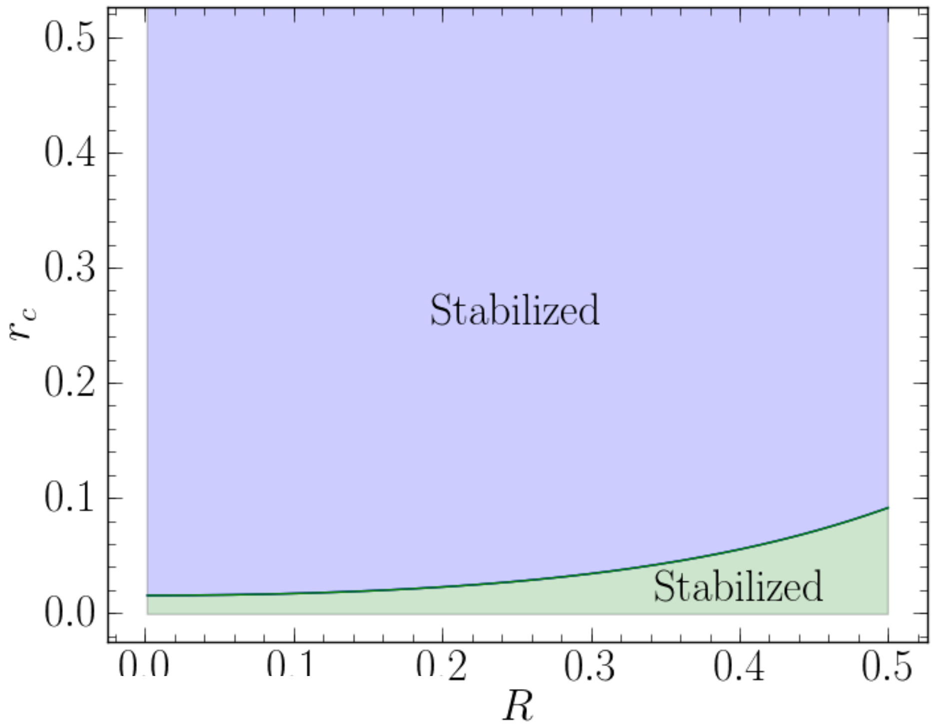
$$\sigma^2 > \frac{G\,M}{r}$$

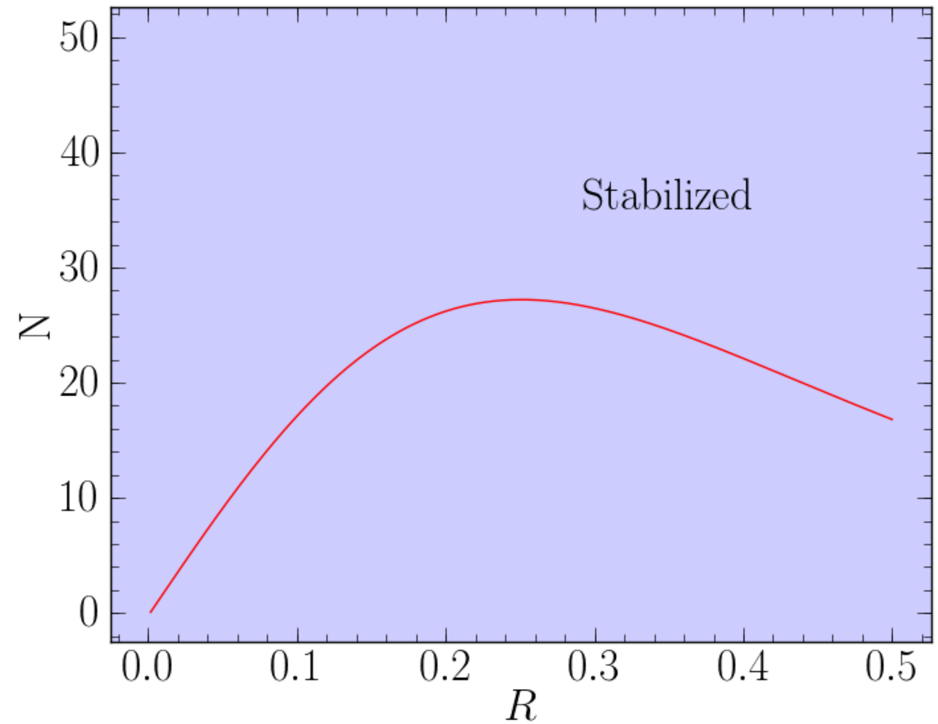
$$\frac{1}{8}\kappa^2\, r^2 > \frac{G\,M}{r}$$

$$\sigma\kappa\, r > \sqrt{8}\frac{G\,M}{r}$$

$$\frac{\sigma\,\kappa}{\pi\, G\,\Sigma} > \sqrt{8}$$

Rotating disks of infinite thickness : Local stability





$$\sigma^2 > \frac{G\,M}{r}$$

$$\frac{1}{8}\kappa^2\, r^2 > \frac{G\,M}{r}$$

$$\sigma \kappa\, r > \sqrt{8}\frac{G\,M}{r}$$

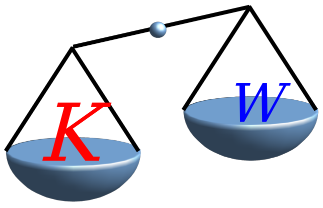
$$\frac{\sigma\,\kappa}{\pi\, G\,\Sigma} > \sqrt{8}$$

Safronov-Toomre critterion

(Safronov 1960, Toomre 1964)

$$Q = \frac{\sigma_R\,\kappa}{3.36\,G\,\Sigma}$$

Disk stability : summary



1. large random motions : σ

→ stabilizes the small scales $\sigma^2 > G\,\Sigma\,\pi\,r$

2. strong differential rotation : κ

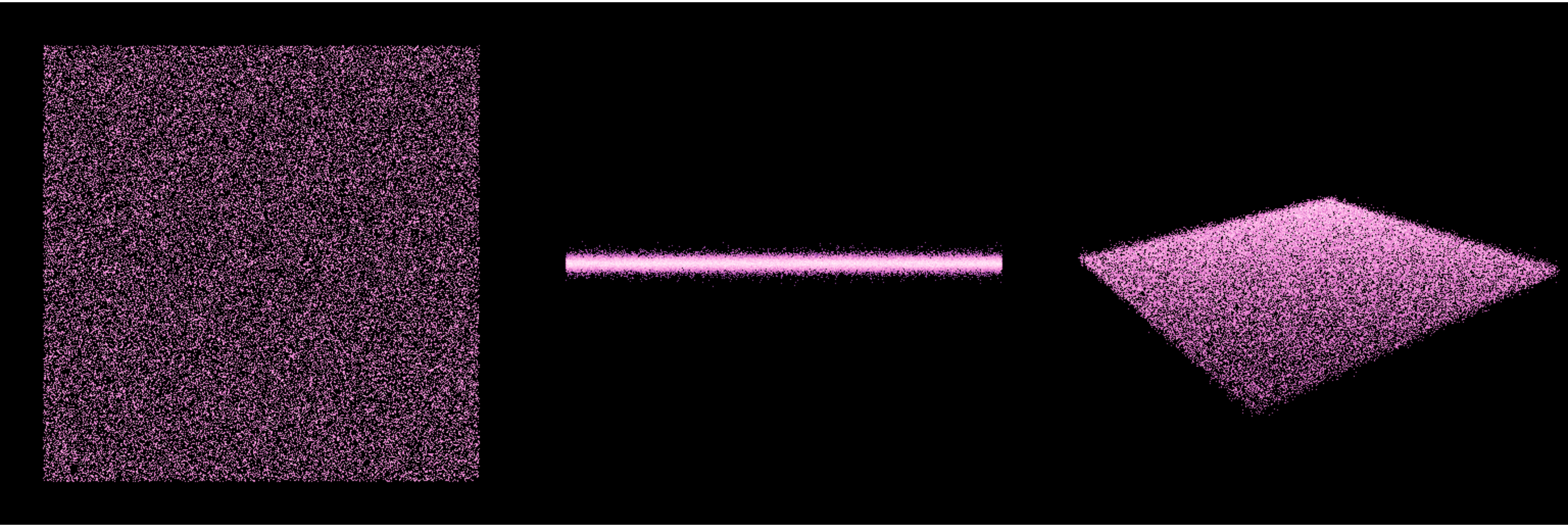
→ stabilizes the large scales $\kappa^2\,r^2 > G\,\Sigma\,\pi\,r$

M74

Nature is always more tricky...

The stability of infinite slab of finite thickness

- isothermal vertical profile



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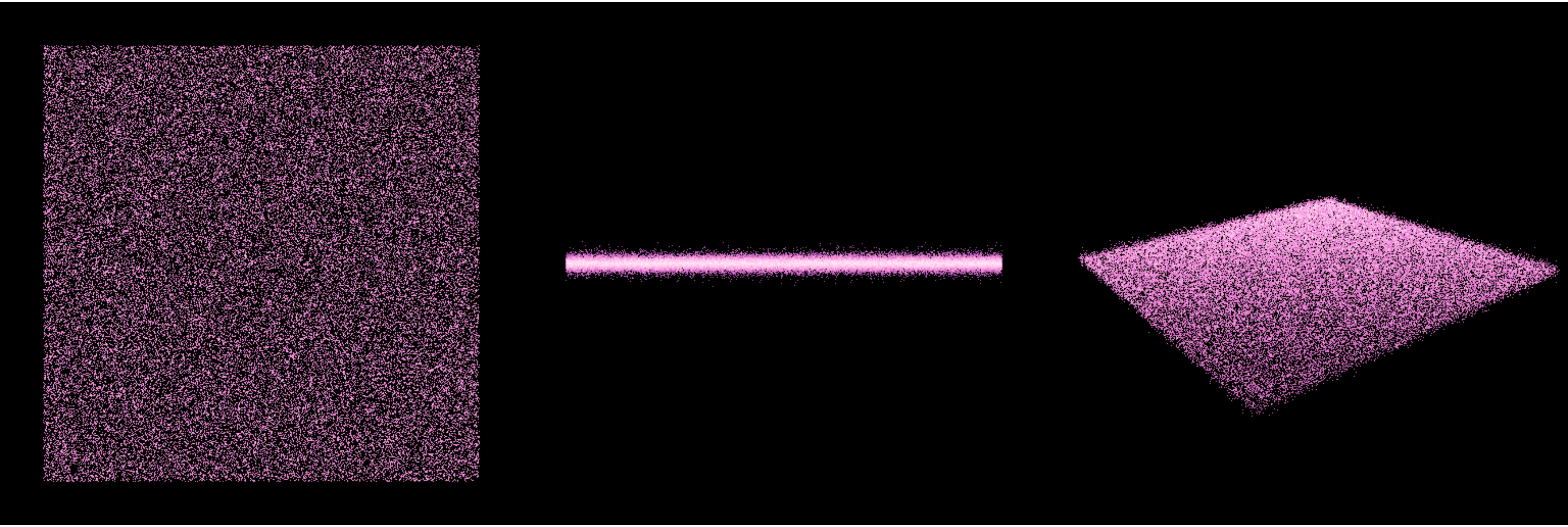


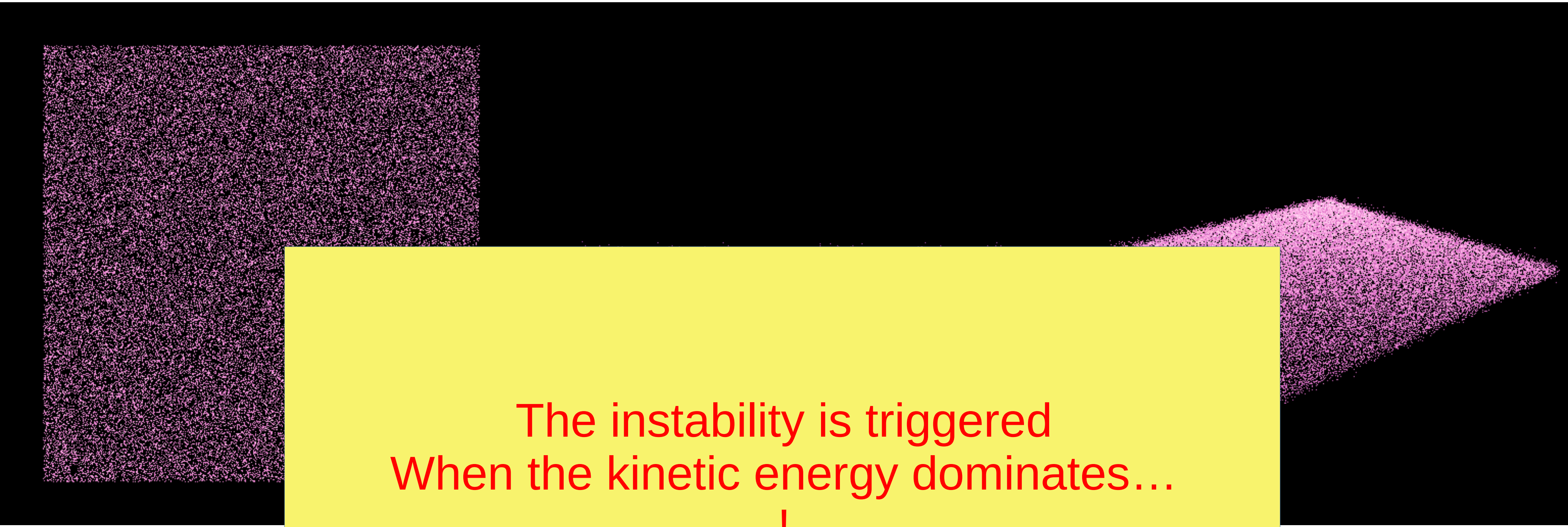
Dispersion relations

horizontal perturbation $$\omega^2 = \kappa^2 - 2\pi G\Sigma k + c_s^2 k^2$$

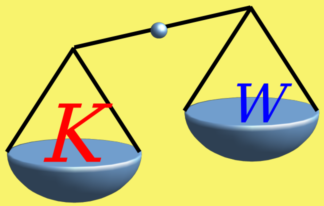
vertical perturbation $$\omega^2 = \nu^2 + 2\pi G\Sigma k - \sigma^2 k^2$$

The stability of infinite slab of <u>finite thickness</u>

• isothermal vertical profile



The instability is triggered
When the kinetic energy dominates…
!

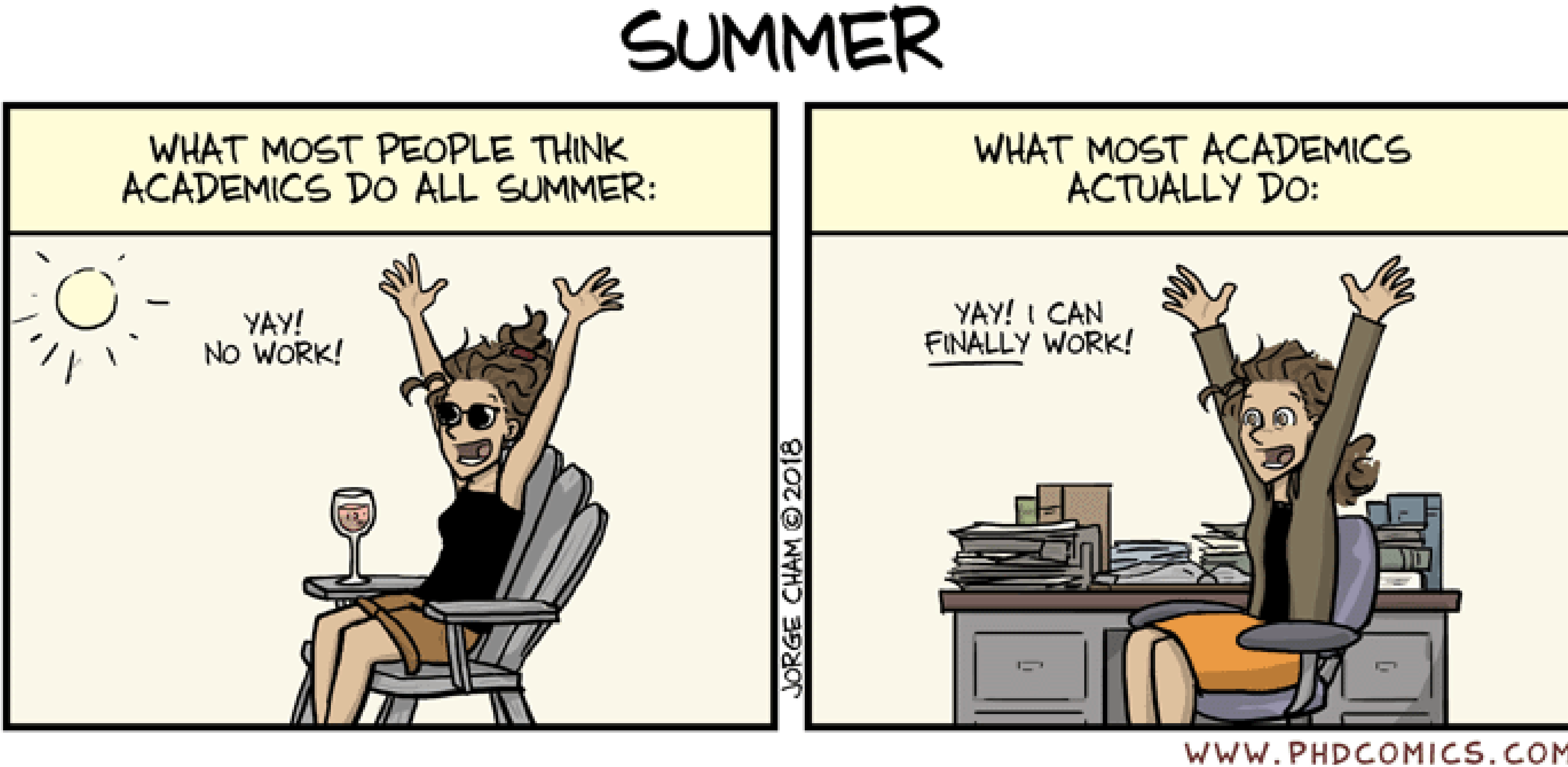


Dispersion re

horizontal pe

vertical pertu

Have a nice break !

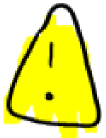


The End

The stability of uniformly rotating systems

(additional material)

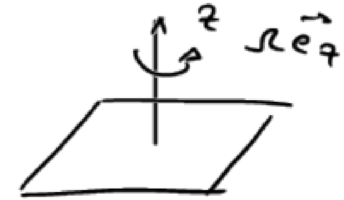
The stability of uniformely rotating systems



- Flattened systems : the geometry is more complex
- Reservoir of kinetic energy (rotation) to feed unstable modes

The uniformly rotating sheet

- infinite disk of zero thickness with surface density Σ_0
    - plane $z = 0$
    - rotation $\vec{\Omega} = \Omega \vec{e}_z$



- 2D perturbation / evolution (no warp, no bending)

① **Continuity equation** (Fluid) $\Sigma_d(x,y,t)$, $\phi(x,y,t)$, $v(x,y,t)$, $p(x,y,t)$

6 inc
6 Equ.

$$\frac{\partial \rho_s}{\partial t} + \vec{\nabla}(\rho_s \vec{v}) = 0 \longrightarrow \boxed{\frac{\partial \Sigma_d}{\partial t} + \vec{\nabla}(\Sigma_d \vec{v}) = 0}$$

with $\vec{v} = \vec{v}(x,y,t) = v_x(x,y,t)\,\vec{e_x} + v_y(x,y,t)\,\vec{e_y}$ (in the plane)

② **Euler Equation**

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho_s} - \vec{\nabla}\phi$$

$$\rightarrow \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\Sigma_d} - \vec{\nabla}\phi \underbrace{- 2\vec{\Omega}\times\vec{v}}_{\text{Coriolis force}} + \underbrace{-\vec{\Omega}\times(\vec{\Omega}\times\vec{x})}_{\text{centrifugal force}}$$

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\Sigma_d} - \vec{\nabla}\phi - 2\Omega(v_x\vec{e_x} + v_y\vec{e_y}) + \Omega^2(x\,\vec{e_x} + y\,\vec{e_y})}$$

③ Poisson equation

$$\nabla^2 \phi = 4\pi G \rho \quad \longrightarrow \quad \boxed{\nabla^2 \phi = 4\pi G \Sigma\, \delta(z)}$$

④ Equation of state (barotropic)

$$p(x,y,t) = p(\Sigma_d(x,y,t))$$

Notes

- sound speed : $v_s^2 = \left.\frac{\partial P(\Sigma)}{\partial \Sigma}\right|_{\Sigma_0}$
- P is the pressure in the plane only $\left[\frac{\text{force}}{\text{lenght}} \right]$
- Σ_d : surf density of the disk only

  Σ = total surface density $\Sigma_d + \underset{\sim}{\Sigma_e}$

  External perturbation

Isolated system at equilibrium Σ_{d0}, ϕ_0, p_0, $\vec{v}_0 = 0$

① **Continuity equation** $\rightarrow$ $0 = 0$

② **Euler Equation** $\vec{\nabla}\phi_0 = \Omega^2(x\,\vec{e}_x + y\,\vec{e}_y)$

F_{grav} $\quad$ $F_{cent.}$

③ **Poisson equation** $\nabla^2\phi_0 = 4\pi G\,\Sigma_{d0}\,\delta(z)$ $\quad$ $(\vec{\nabla} p_0 = 0)$

Note by symmetry, $\vec{\nabla}\phi_0 \parallel \vec{e}_z$

$$\Rightarrow \quad \phi_0 = 2\pi G\,\Sigma_0 |z|$$

poisson $\Rightarrow$ $\Sigma_{d0} = 0$

(need a Jeans swindle

consider only Poisson

for the perturbed part)

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla}\phi_e$

$$\Sigma_{d0} \rightarrow \Sigma_{d0} + \varepsilon \Sigma_{d1}(x,y,t)$$

$$\vec{V}_0 = 0 \rightarrow \varepsilon V_1(x,y,t)$$

$$\Sigma_0 \rightarrow \Sigma_0 + \varepsilon \Sigma_1(x,y,t) \quad \text{total density}$$

$$\phi_0 \rightarrow \phi_0 + \varepsilon \phi_1(x,y,t) \quad \text{total potential}$$

$$= \varepsilon \phi_{d1}(x,y,t) + \varepsilon \phi_e(x,y,t)$$

A first order in ε, we get

① Continuity equation

$$\frac{\partial}{\partial t}\Sigma_{d1} + \Sigma_0 \vec{\nabla}\vec{V}_1 = 0$$

② Euler Equation

$$\frac{\partial}{\partial t}\vec{V}_1 = -\frac{v_s^2}{\Sigma_0}\vec{\nabla}\Sigma_{d1} - \vec{\nabla}\phi_1 - 2\vec{\Omega}\times\vec{V}_1$$

contribution of the rotation

the rest is similar to the homogeneous case

③ Poisson equation

$$\nabla^2\phi_1 = 4\pi G \Sigma_1 \delta(z)$$

④ "Equation of state"

$$v_s^2 = \left.\frac{\partial p}{\partial \Sigma}\right|_{\Sigma_0}$$

Solution

$\frac{\partial}{\partial t}$ ① : $\frac{\partial^2}{\partial t^2}\Sigma_{d1} + \Sigma_0 \vec{\nabla} \frac{\partial}{\partial t}\vec{v}_1 = 0$

$\vec{\nabla}\cdot$ ② : $\vec{\nabla}\cdot\frac{\partial}{\partial t}\vec{v}_1 = -\frac{v_s^2}{\Sigma_0}\nabla^2\Sigma_{d1} - \nabla^2\phi_1 + 2\Omega\vec{\nabla}\vec{v}_1$

we get

$$\frac{\partial^2}{\partial t^2}\Sigma_{d1} - \underbrace{v_s^2\nabla^2\Sigma_{d1}}_{\text{pressure}} - \underbrace{\Sigma_0\nabla^2\phi_1}_{\text{gravity}} + \underbrace{2\Sigma_0\Omega\vec{\nabla}\vec{v}_1}_{\text{cent. force}} = 0$$

Assume solutions of the form (waves) with $\vec{k} = k\vec{e}_x$

(no loss of generality isotropy)

$$\begin{cases} \Sigma_1(x,y,t) = \Sigma_a e^{i(\vec{k}\vec{x}-\omega t)} \\ \Sigma_{d1}(x,y,t) = \Sigma_{da} e^{i(\vec{k}\vec{x}-\omega t)} \\ \vec{v}_1(x,y,t) = (v_{ax}\vec{e}_x + v_{ay}\vec{e}_y)\, e^{i(\vec{k}\vec{x}-\omega t)} \\ \phi_1(x,y,z=0,t) = \phi_a e^{i(\vec{k}\vec{x}-\omega t)} \end{cases}$$

(! here, in the plane, $z=0$)

③ Poisson equation $\qquad \nabla^2 \phi_1 = 4\pi G \Sigma_1 \, \delta(z)$

We want $\phi_1(x,y,z)$, such that

$$\begin{cases} \bullet\ z = 0 : & \phi_1(x,y,z=0,t) = \phi_a \, e^{i(kx - \omega t)} \\ \bullet\ z \neq 0 : & \nabla^2 \phi_1 = 0 \end{cases}$$

$$\Rightarrow \quad \phi_1(x,y,z,t) = \phi_a \, e^{i(kx - \omega t) - |kz|}$$

Solving Poisson links Σ_a with $\phi_a = -\dfrac{2\pi G \Sigma_a}{|k|}$

$$\Rightarrow \quad \phi_1(x,y,z,t) = -\frac{2\pi G \Sigma_a}{|k|} \, e^{i(kx - \omega t) - |kz|}$$

Introducing $\Sigma_1(x,y,t)$, $\Sigma_{d1}(x,y,t)$, $\vec{v}_1(x,y,t)$, $\phi_1(x,y,z=0,t)$

in the linearized equations, we get the dispersion relation

$$\omega^2 = v_s^2 k^2 - 2\pi G|k|\Sigma_0 \frac{\Sigma_a}{\Sigma_{da}} + 4\Omega^2 = 0$$

Note that $\Sigma_a = \Sigma_{da} + \Sigma_e$

If we consider the evolution without the perturbation $\Sigma_e = 0$

$$\omega^2 = v_s^2 k^2 - 2\pi G|k|\Sigma_0 + 4\Omega^2$$

Interpretation

$$\omega^2 = v_s^2 k^2 - 2\pi G|k|\Sigma_0 + 4\Omega^2$$

$\omega^2 > 0$ STABLE

$\omega^2 < 0$ UNSTABLE

$\Rightarrow$ Ω helps to stabilize the slab

① $\Omega^2 = 0$ no rotation

$$\omega^2 = v_s^2 k^2 - 2\pi G|\vec{k}|\Sigma_0$$

$$= v_s^2 (k^2 - k_J|\vec{k}|)$$

$$k_J := \frac{2\pi G\Sigma_0}{v_s^2}$$

STABLE if $|\vec{k}| > k_J$

UNSTABLE if $|\vec{k}| < k_J$

homogeneous system

$$\omega^2 = v_s^2 k^2 - 4\pi G\rho_0$$

$$= v_s^2 (k^2 - k_J^2)$$

$$k_J^2 := \frac{4\pi G\rho_0}{v_s^2}$$

② $\Omega \neq 0 \ , \ V_s = 0$ no pressure

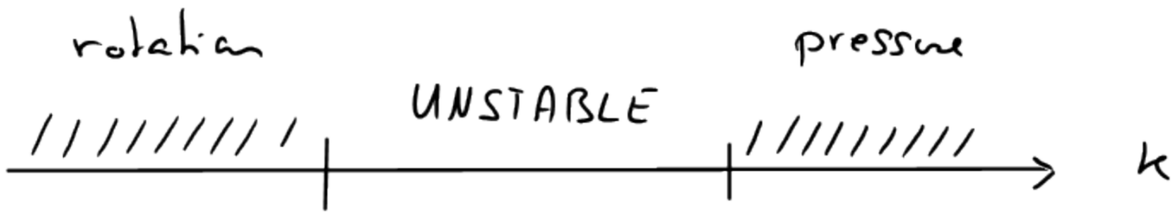
$$\omega^2 = 4\Omega^2 - 2\pi G |\vec{k}| \Sigma_0$$

STABLE if $|\vec{k}| < \dfrac{2\Omega^2}{\pi G \Sigma_0}$ (at large scale)

UNSTABLE if $|\vec{k}| > \dfrac{2\Omega^2}{\pi G \Sigma_0}$ (at small scale)

$\Rightarrow$ nothing can prevent the disk to be unstable at small scale

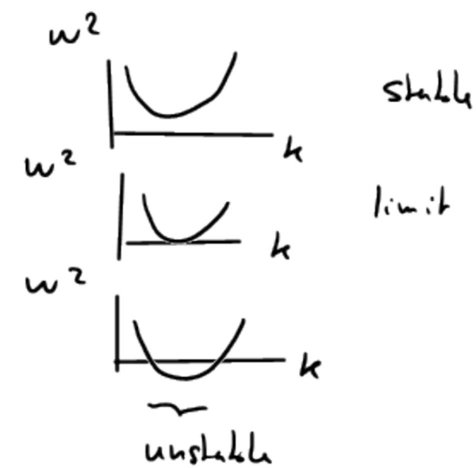
③ Complete stability



$$k_1 = \frac{2\Omega^2}{\pi G \Sigma_0} \qquad k_2 = \frac{2\pi G \Sigma_0}{v_s^2}$$

if $k_1 \geq k_2 \Rightarrow$ complete stability ! $\quad \dfrac{\Omega v_s}{G\Sigma_0} \geq \pi$

More precisely



$$\omega^2 = v_s^2 k^2 - 2\pi G |k| \Sigma_0 + 4\Omega^2 = 0$$

$$\boxed{\frac{v_s \Omega}{G\Sigma_0} \geq \frac{1}{2}\pi} \approx 1.57$$

The uniformly rotating stellar sheet

- if the velocity DF is Maxwellian

$$f_0(\vec{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} e^{-\frac{v^2}{2\sigma^2}}$$

($\vec{v}$ in the plane)

- The stability criterion becomes

$$\boxed{\frac{\sigma\Omega}{G\Sigma_0} \gtrsim 1.68}$$

hydro

$$\boxed{\frac{v_s\Omega}{G\Sigma_0} \gtrsim \frac{1}{2}\pi} \simeq 1.57$$

The stability of rotating disks :

the dispersion relation

(additional material)

The dispersion relation for a razor thin fluid disk

polar coordinates

$$\Sigma_d(R,\phi)\,,\quad \phi_d(R,\phi)\,,\quad p(R,\phi)\,,\quad \vec{v}(R,\phi) = v_R(R,\phi)\,\vec{e}_R + v_\phi(R,\phi)\,\vec{e}_\phi$$

① **Continuity equation**

$$\frac{\partial \Sigma_d}{\partial t} + \vec{\nabla}(\Sigma_d \vec{v}) = 0$$

$$\frac{\partial \Sigma_d}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\,\Sigma_d v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\Sigma_d v_\phi)$$

② **Euler equation**

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\Sigma_d} - \vec{\nabla}\phi$$

$$\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial \phi}{\partial R} - \frac{1}{\Sigma_d}\frac{\partial p}{\partial R}$$

$$\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R}\frac{\partial \phi}{\partial \phi} - \frac{1}{\Sigma_d}\frac{1}{R}\frac{\partial p}{\partial \phi}$$

③ Poisson $\nabla^2 \phi = 4\pi G \Sigma \delta(z)$

④ Equation of state (polytropic)

$p = K \Sigma_d^{\gamma}$ $\quad v_s^2 = \gamma K \Sigma_0^{\gamma-1}$ (unperturbed sound speed)

$h = \frac{\gamma}{\gamma-1} K \Sigma_d^{\gamma-1}$ $\quad \frac{\partial h}{\partial R} = \gamma K \Sigma_d^{\gamma-2} \frac{\partial \Sigma}{\partial R}$

(Specific enthalpy) $\quad \frac{\partial h}{\partial \phi} = \gamma K \Sigma_d^{\gamma-2} \frac{\partial \Sigma}{\partial \phi}$

<u>The Euler Equation becomes</u>

$$\begin{cases} \frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial}{\partial R}(\phi + h) \\ \frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R} \frac{\partial}{\partial \phi}(\phi + h) \end{cases}$$

Isolated system at equilibrium

Σ_{d0}, ϕ_0, p_0, $\vec{v}_0$ + axisymmetric + $v_r = 0$ $\left(\frac{\partial}{\partial \phi} = 0\right)$

① Continuity equation

$0 = 0$

② Euler equation

ϕ: $0 = 0$

R:

$$\frac{v_{\phi 0}^2}{R} = \frac{\partial}{\partial R}(\phi_0 + h_0) = \frac{\partial \phi_0}{\partial R} + \underbrace{v_s^2 \frac{d}{dR} \ln \Sigma_0}_{(*)}$$

centripetal acceleration — gravity force — pressure force

Notes

① (*) $v_s^2 \frac{d}{dR} \ln \Sigma_0 = \left.\frac{\partial p}{\partial \Sigma}\right|_{\Sigma_0} \frac{1}{\Sigma_0} \frac{\partial \Sigma_0}{\partial R} = \frac{1}{\Sigma_0} \frac{\partial p}{\partial R} = \frac{1}{\Sigma_0} \vec{\nabla} p$

② $v_s \simeq 10$ km/s, $v_\phi \simeq 100$ km/s $\Big\}$ ⇒ we neglect the pressure

③ $v_{\phi 0} \simeq \sqrt{R \frac{\partial \phi_0}{\partial R}} = R\, \Omega(R)$

The response of the system to a weak perturbation $-\varepsilon \vec{\nabla}\phi_e$

$$\Sigma_{d0} \rightarrow \Sigma_{d0} + \varepsilon\,\Sigma_{d1}$$

$$v_{R_0} = 0 \rightarrow \varepsilon\, v_{R1}$$

$$v_{\phi_0} \rightarrow v_{\phi_0} + \varepsilon\, v_{\phi_1}$$

$$h_0 \rightarrow h_0 + \varepsilon\, h_1$$

$$\phi_{d0} \rightarrow \phi_{d0} + \varepsilon\,\phi_{d1}$$

disk only without the perturbation

$$\phi_{d0} = \phi_0 \rightarrow \phi_{d0} + \underbrace{\varepsilon\,\phi_1}_{\varepsilon\,\phi_{d1} + \varepsilon\,\phi_e}$$

total potential, includes the perturbation

Linearized equations for a razor thin fluid disk

① Continuity equation

$$\frac{\partial}{\partial t}\Sigma_{d1} + \Omega\frac{\partial \Sigma_{d1}}{\partial \phi} + \frac{1}{R}\frac{\partial}{\partial R}\left(R v_{R1}\Sigma_0\right) + \frac{\Sigma_0}{R}\frac{\partial v_{\phi 1}}{\partial \phi} = 0$$

② Euler equation

$$\frac{\partial v_{R1}}{\partial t} + \Omega\frac{\partial v_{R1}}{\partial \phi} - 2\Omega v_{\phi 1} = -\frac{\partial}{\partial R}\left(\Phi_1 + h_1\right)$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \underbrace{\left[\frac{\partial}{\partial R}(\Omega R) + \Omega\right]}_{-2B(R)} v_{R1} + \Omega\frac{\partial v_{\phi 1}}{\partial \phi} = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(\Phi_1 + h_1\right)$$

$-2B(R)$ Oort constant $\qquad \kappa^2 = -4B\Omega$

Solutions

assume waves of the form (can be a spiral)

$$\Sigma_{d1} = Re\left[\Sigma_{da}(R)\, e^{i(m\phi - \omega t)}\right]$$
$$v_{R1} = Re\left[v_{Ra}(R)\, e^{i(m\phi - \omega t)}\right]$$
$$v_{\phi 1} = Re\left[v_{\phi a}(R)\, e^{i(m\phi - \omega t)}\right]$$
$$h_1 = Re\left[h_a(R)\, e^{i(m\phi - \omega t)}\right]$$

} without perturbation

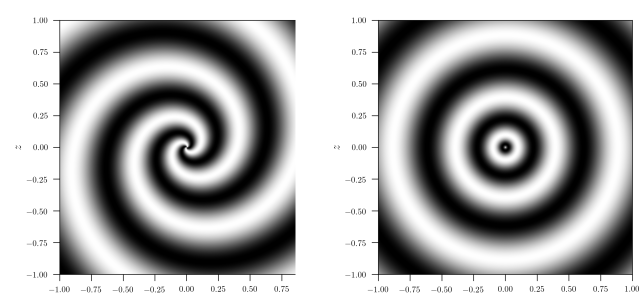
5 radial functions

$$\Sigma_1 = Re\left[\Sigma_a(R)\, e^{i(m\phi - \omega t}\right]$$
$$\phi_1 = Re\left[\phi_a(R)\, e^{i(m\phi - \omega t}\right]$$

} total: disk + perturbation

$$\Sigma_a = \Sigma_{da} + \Sigma_e$$
$$\phi_a = \phi_{da} + \phi_e$$

$$\Sigma_{d1}(R,\phi,t) = Re\left[H_{da}(R,t)e^{i(m\phi + f(R,t) - \omega t)}\right]$$



① The continuity equation gives

$$-i(\omega - m\Omega)\Sigma_{da} + \frac{1}{R}\frac{d}{dR}\left(R\, v_{Ra}\, \Sigma_0\right) + \frac{im\Sigma_0}{R} v_{\phi a} = 0$$

② The Euler equation gives

$$\begin{cases} v_{Ra}(R) = \dfrac{i}{\Delta}\left[(\omega - m\Omega)\dfrac{d}{dR}(\phi_a + h_a) - \dfrac{2m\Omega}{R}(\phi_a + h_a) \right] \\ v_{\phi a}(R) = -\dfrac{i}{\Delta}\left[2B\dfrac{d}{dR}(\phi_a + h_a) + \dfrac{m(\omega - m\Omega)}{R}(\phi_a + h_a) \right] \end{cases}$$

with $\Delta = \kappa^2 - (\omega - m\Omega)^2$

Notes

① v_{Ra}, $v_{\phi a}$, κ, Δ, Ω are functions of R only

② if $\Delta = 0 \Rightarrow$ divergence

$\Omega_p = \dfrac{\omega}{m}$ pattern speed

$$\kappa^2 = (\omega - m\Omega)^2$$

$$= m^2\left(\frac{\omega}{m} - \Omega\right)^2 = m^2(\Omega_p - \Omega)^2$$

$$\Omega_p = \Omega \pm \frac{\kappa}{m}$$

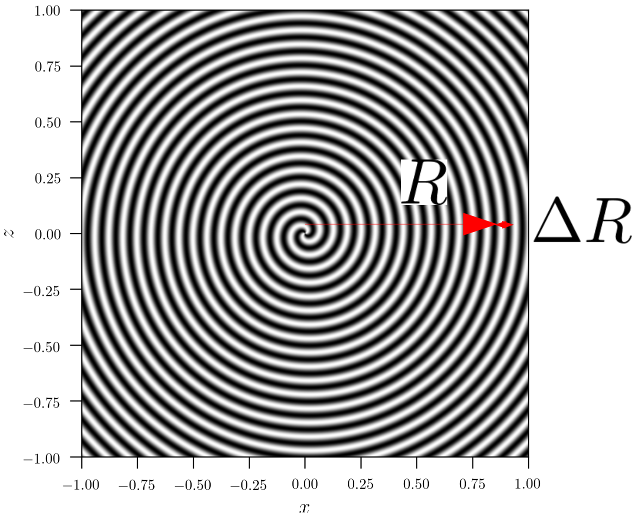
!!! problems at the Lindblad resonance

④ "Equation of state" $h = \frac{\gamma}{\gamma - 1} K \Sigma_d^{\gamma - 1}$ $v_s^2 = \gamma K \Sigma_0^{\gamma - 1}$

$$h_a = \gamma K \Sigma_0^{\gamma - 2} \Sigma_{da} = v_s^2 \frac{\Sigma_{da}}{\Sigma_0}$$

=> we have 4 equations for 5 unknowns $\Sigma_{da}, h_a, \phi_a, v_{Ra}, v_{\phi a}$

⑤ Poisson Equation



Stability of collisionless systems

The WKB approximation

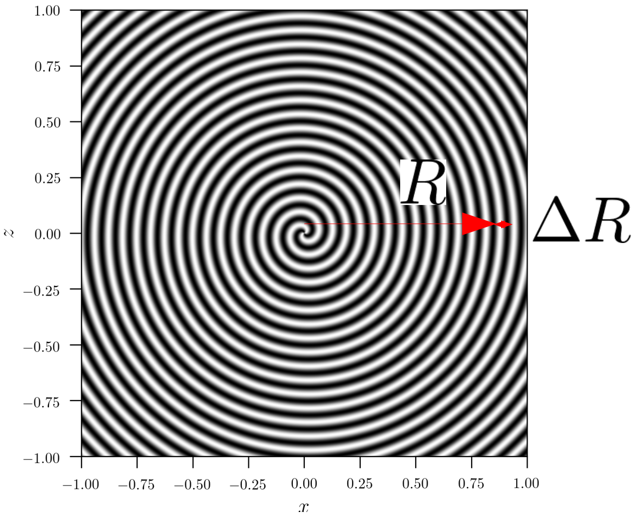
Potential of tightly wound spiral pattern

Tightly wound spiral approximation

We assume that $\underline{\Delta R \ll R}$

but $\Delta R = \frac{2\pi}{m|k|} \sim \frac{1}{|k|}$

Thus : $\boxed{|kR| \gg 1}$



WKB approximation

(Wentzel - Kramers - Brillouin)

Model of a spiral surface density

$$\Sigma_d(R,\phi,t) = \underbrace{\Sigma_{d0}(R)}_{\text{at equilibrium}} + \varepsilon \underbrace{\Sigma_{d1}(R,\phi,t)}_{\text{non axi-symmetric perturbation}}$$

(spiral pattern)

Lets formulate Σ_1 as

shape function

$$\Sigma_1(R,\phi,t) = \mathrm{Re}\left(\underbrace{H(R,t)}_{\text{"slow" radial variation}} \underbrace{e^{i(m\phi + f(R,t))}}_{\text{"rapid" variation}} \right)$$

amplitude of the spiral arm

What is the corresponding potential ?

"rapid" radial variation of $\Sigma(R)$

$\Rightarrow$ the contribution of the distant parts cancels

$\Rightarrow$ the potential is determined from $\Sigma_1(R,\phi,t)$ within a few k

<u>Method</u> : expand the shape function around $(\underline{R_0, \phi_0})$ any "random point"

$$
\begin{cases}
\bullet\ \ f(R,t) \simeq f(R_0,t) + \dfrac{\partial f}{\partial R}(R-R_0) \\
\qquad\qquad = f(R_0,t) + k(R_0,t)(R-R_0) \qquad k = |\vec{k}| \\
\bullet\ \ \mathcal{H}(R,t) \simeq \mathcal{H}(R_0,t)
\end{cases}
$$

<u>We get</u> :

$$k = k(R_0, t)$$

$$\Sigma_n(R,\phi,t) \;\tilde{=}\; H(R_0,t)\, e^{i\left(m\phi_0 + g(R_0,t) + k(R-R_0)\right)}$$

$$= H(R_0,t)\, e^{i(m\phi_0 + g(R_0,t))}\; e^{ik(R-R_0)}$$

$$= \underbrace{\Sigma_a}_{\text{no radial dependency}}\; e^{ik(R-R_0)}$$

this is a plane wave with $\vec{k} = k\,\vec{e}_R$ (see the rotating sheet)

<u>The corresponding potential is thus</u>

$$\phi_n(R,\phi,z,t) \;\tilde{=}\; -\frac{2\pi G \Sigma_a}{k}\, e^{ik(R-R_0) - |kz|}$$

$$\tilde{=}\; -\frac{2\pi G}{k}\, \underbrace{H(R_0,t)\, e^{i(m\phi_0 + g(R_0,t))}}_{\Sigma_a}\; e^{ik(R-R_0) - |kz|}$$

We are free to set $R = R_0$, $\phi = \phi_0$, $z = 0$ and thus, we get

$$\Phi_n(R, \phi, z, t) = -\frac{2\pi G}{k} H(R,t)\, e^{i[m\phi + f(R,t)]}$$

Stability of collisionless systems

Back to the dispersion relation

④ "Equation of state"

$$h = \frac{\gamma}{\gamma-1} K \Sigma_d^{\gamma-1} \qquad v_s^2 = \gamma K \Sigma_0^{\gamma-1}$$

$$\boxed{h_a = \gamma K \Sigma_0^{\gamma-2} \Sigma_{da} = v_s^2 \frac{\Sigma_{da}}{\Sigma_0}}$$

$\Rightarrow$ we have 4 equations for 5 unknowns $\Sigma_{da}, h_a, \phi_a, v_{Ra}, v_{\phi a}$

⑤ Poisson Equation use WKB approximation

$$\Sigma_1 = R_e\left[\Sigma_a(R)\, e^{i(m\phi - \omega t)}\right]$$

$$= R_e\left[e^{-i\omega t}\, \Sigma_a(R)\, e^{im\phi}\right]$$

$$= R_e\left[H(R,t)\, e^{i f(R,t)}\, e^{im\phi}\right]$$

$$\phi_1 = -\frac{2\pi G}{k} \underbrace{H(R,t)}_{\text{slow variation}} \underbrace{e^{i f(R,t)}}_{\text{fast variation}} \underbrace{e^{im\phi}}_{\text{azimuthal symmetry}}$$

$$\boxed{k = \frac{\partial f}{\partial R} \qquad |kR| \gg 1}$$

In addition

(4) " Eq. of state " $\Rightarrow$ $h_a \sim \Sigma_{da} \sim \phi_1 \sim e^{if(R,t)}$

Thus a) $\frac{d}{dR}(\phi_a + h_a) \sim (\phi_a + h_a)\, i \frac{d}{dR} f = (\phi_a + h_a)\, ik$

b) $\frac{1}{R}(\phi_a + h_a) \cong \frac{1}{R}(\phi_a + h_a)\, ik \frac{1}{ik} = \frac{-i}{Rk} \frac{d}{dR}(\phi_a + h_a)$

as $Rk \gg 1$ $\quad \frac{1}{R}(\phi_a + h_a) \ll \frac{d}{dR}(\phi_a + h_a)$

(1) The continuity equation becomes $\quad \frac{d}{dR}(R\Sigma_0 V_{Ra}) = ikR\Sigma_0 V_{Ra}$

$$-(\omega - m\Omega)\,\Sigma_{da} + k\,\Sigma_0 V_{Ra} = 0$$

(2) The Euler equation becomes $\quad$ + EoS $(h_a \sim \Sigma_{da})$ + poisson $(\Sigma_a \sim \phi_a)$

$$V_{Ra} = -\frac{\omega - m\Omega}{\Delta} k(\phi_a + h_a)$$

$$V_{\phi a} = -\frac{2iB}{\Delta} k(\phi_a + h_a)$$

We can solve to get

$$\Sigma_{da} = \left(\frac{2\pi G \Sigma_0 |k|}{\kappa^2 - (\omega - m\Omega)^2 + v_s^2 k^2} \right) \Sigma_a$$

Which gives the dispersion relation if $\Sigma_e = 0$ $\quad (\Sigma_{da} = \Sigma_a)$

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

<u>Note:</u> if $\Omega(R) = \text{cte}$ $\quad \kappa = 2\Omega$ and we recover the dispersion relation for uniformly rotating sheet

$$\omega^2 = 4\Omega^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

Interpretation

Axisymmetric perturbations $\quad m = 0$

"Cold disk" $\quad v_s = 0$

$$\omega^2 = \kappa^2 - 2\pi G \Sigma_0 |k|$$

$$k_{crit} := \frac{\kappa^2}{2\pi G \Sigma_0} \qquad \lambda_{crit} := \frac{2\pi}{k_{crit}} = \frac{4\pi^2 G \Sigma_0}{\kappa^2}$$

$|k| > k_{crit} \quad (\lambda < \lambda_{crit}) \quad \omega^2 < 0$ — UNSTABLE

$|k| < k_{crit} \quad (\lambda > \lambda_{crit}) \quad \omega^2 > 0$ — STABLE

rotation stabilizes at large scale

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

"Non rotating fluid disk"

$\kappa = 0$

⚠ not realistic

$$\omega^2 = -2\pi G \Sigma_0 |k| + v_s^2 k^2$$

$$k_{crit} := \frac{2\pi G \Sigma_0}{v_s^2} \quad (= k_{Jeans}) \qquad \lambda_{crit} = \frac{2\pi}{k_{crit}} = \frac{v_s^2}{G \Sigma_0}$$

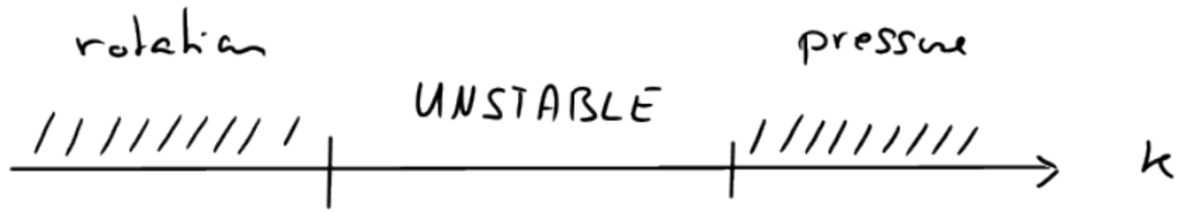
$\lvert k\rvert > k_{crit}$	$(\lambda < \lambda_{crit})$	$\omega^2 > 0$	**STABLE**
$\lvert k\rvert < k_{crit}$	$(\lambda > \lambda_{crit})$	$\omega^2 < 0$	**UNSTABLE**

pressure stabilizes the disk

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

Rotating fluid disk

$$\omega^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$



$$k_1 = \frac{2\kappa^2}{\pi G \Sigma_0} \qquad k_2 = \frac{2\pi G \Sigma_0}{v_s^2}$$

Complete stability

$$k_1 \gg k_2$$

$$Q := \frac{\kappa v_s}{\pi G \Sigma_0} > 1$$

Stability of stellar disks

$$Q := \frac{\varkappa v_s}{\pi G \Sigma_0} > 1$$

$$Q := \frac{\varkappa \sigma_R}{3.36\, G \Sigma_0} > 1$$

Safronov - Toomre criterion

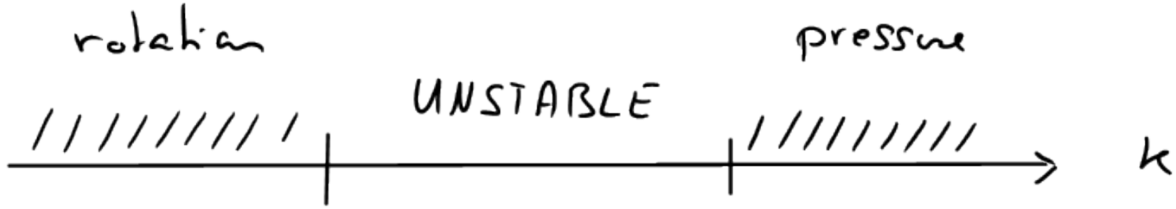
Non axisymmetric perturbations ①

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2$$

$$\omega = \sqrt{\kappa^2 - 2\pi G \Sigma_0 |k| + v_s^2 k^2} + m\Omega$$

- the stability is determined for $m = 0$ (value of Γ)
- $m\Omega \in \mathbb{R}_e$ → add an oscillatory term $e^{-i m\Omega}$ with a frequency that correspond to the passage of spiral arm (ex: if $\Gamma = 0$)

Non axisymmetric perturbations ②



$$k_1 = \frac{\kappa^2}{2\pi G \Sigma_0} \qquad k_2 = \frac{2\pi G \Sigma_0}{v_s^2}$$

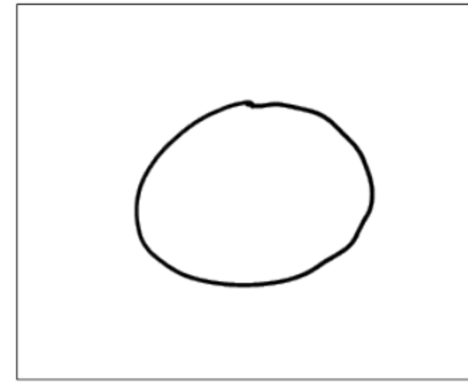
- $\lambda_1 = \frac{2\pi}{k_1} = 4\pi^2 \frac{G\Sigma_0}{\kappa^2}$ = "max size" of a "clump"
- number of "clumps" $\quad \frac{2\pi R}{\lambda_1} = R \cdot k = \frac{\kappa^2 R}{2\pi G \Sigma_0} = m$

→ number of spiral arms

$$X = \frac{\kappa^2 R}{2\pi G \Sigma m} \geqslant 3$$

↑
ensure the stability of mode m

Physical interpretation of the Jeans instability



Specific potential energy of an homogeneous sphere of radius r

$$|W| \cong \frac{G M(r)}{r} \qquad M(r) = \frac{4}{3}\pi r^3 \rho_0$$

$$\cong \frac{4}{3}\pi r^2 \rho_0 G$$

$r > r_J \qquad |W| > 2T$

UNSTABLE

$r < r_J \qquad |W| < 2T$

STABLE

Specific kinetic energy of an homogeneous sphere of radius r

$$K \cong \frac{1}{2}\sigma^2$$

Radius for which we have the virial equilibrium $\quad 2T = |W|$

$$r = \sqrt{\frac{3\sigma^2}{4 G \rho_0 \pi}} \qquad \sim \qquad \lambda_J = \sqrt{\frac{\pi V_s^2}{G \rho_0}}$$