Astrophysics IV, Dr. Yves Revaz EPFL

4th year physics Exercises week 13

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Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

We start with the collisionless Boltzmann equation and Poisson's equation under the form:

$$\frac{\partial f}{\partial t} + [f, H] = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{1}$$

$$\nabla^2 \Phi_s(\mathbf{x}, t) = 4\pi G \int \mathrm{d}^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t) \quad , \tag{2}$$

From $\Phi = \Phi_0 + \varepsilon\, \Phi_1$, the hamiltonian $H = \frac{1}{2}v^2 + \Phi$ becomes

$$H = H_0 + \varepsilon\, \Phi_1 \quad \text{with} \quad H_0 = \frac{1}{2}v^2 + \Phi_0$$

Introducing $f = f_0 + \varepsilon\, f_1$, $H = H_0 + H\varepsilon\, \Phi_1$ and $\Phi = \Phi_0 + \varepsilon\, \Phi_1$ in 1, we get

$$\begin{aligned}
\frac{\partial f_0}{\partial t} + \varepsilon\, \frac{\partial f_1}{\partial t} + \underbrace{[f_0, H_0]}_{0} + \varepsilon\, [f_1, H_0] + \underbrace{\varepsilon^2\, [f_1, \Phi_1]}_{negligible} + \varepsilon\, [f_0, \Phi_1] &= 0 \\
\Rightarrow \quad \frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_1] &= 0
\end{aligned}$$

Introducing $\Phi_s = \Phi_0 + \epsilon \Phi_{s1}$ and $f = f_0 + \varepsilon\, f_1$ in 2, we obtain:

$$\begin{aligned}
\nabla^2\Phi_0 + \varepsilon\, \nabla^2 \Phi_{s1} &= 4\pi G \left[\int \mathrm{d}^3\mathbf{v}\, f_0 + \varepsilon \int \mathrm{d}^3\mathbf{v}\, f_1\right] \\
\Rightarrow \quad \nabla^2 \Phi_{s1} &= 4\pi G \int \mathrm{d}^3\mathbf{v}\, f_1 \quad \text{because} \quad \nabla^2 \Phi_0 = 4\pi G \int \mathrm{d}^3\mathbf{v}\, f_0
\end{aligned}$$

It may appear strange, at first sight, that we introduce Φ_s instead of the total potential Φ into 2; this is because we are interested only in the density of the system considered *alone*, not in the density causing the perturbing potential. In other words, we consider the system as isolated, except for the perturbation which is caused by some other system, the properties of which (extent, density etc.) do not matter here.

Problem 2:

Introducing the quantities (14) of the énoncé into the continuity equation, we obtain:

$$\begin{aligned}
\frac{\partial \rho_0}{\partial t} + \varepsilon\, \frac{\partial \rho_{s1}}{\partial t} + \nabla \cdot \left[(\rho_0 + \varepsilon\, \rho_{s1})\,(\mathbf{v}_0 + \varepsilon\, \mathbf{v}_1)\right] &= 0 \\
\underbrace{\frac{\partial \rho_0}{\partial t} + \nabla\left(\rho_0 \mathbf{v}_0\right)}_{0} + \varepsilon\, \frac{\partial \rho_{s1}}{\partial t} + \varepsilon\, \left[\nabla\left(\rho_{s1}\mathbf{v}_0\right) + \nabla\left(\rho_0 \mathbf{v}_1\right)\right] + \mathcal{O}\left(\varepsilon^2\right) &= 0 \\
\Rightarrow \quad \frac{\partial \rho_{s1}}{\partial t} + \nabla\left(\rho_0 \mathbf{v}_1\right) + \nabla\left(\rho_{s1} \mathbf{v}_0\right) &= 0
\end{aligned}$$

Similarly, for Euler's equation:

$$\begin{aligned}
\frac{\partial \mathbf{v}_0}{\partial t} + \varepsilon \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\,\mathbf{v}_0 + \varepsilon\,(\mathbf{v}_1 \cdot \nabla)\,\mathbf{v}_0 + \varepsilon\,(\mathbf{v}_0 \cdot \nabla)\,\mathbf{v}_1 + \mathcal{O}\left(\varepsilon^2\right) &= -\nabla\left(h_0 + \varepsilon h_1 + \Phi_0 + \varepsilon \Phi_1\right) \\
\Rightarrow \quad \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\,\mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla)\,\mathbf{v}_0 = -\nabla\left(h_1 + \Phi_1\right) &= -\nabla\left(h_1 + \Phi_{s1} + \Phi_e\right)
\end{aligned}$$

because the 0^{th} order terms cancel out. For Poisson's equation, we have

$$\begin{aligned}
\nabla^2 \Phi_s &= 4\pi G\,\rho_s \\
\nabla^2 \Phi_0 + \varepsilon\,\nabla^2 \Phi_{s1} &= 4\pi G\,\left(\rho_s + \varepsilon\,\rho_{s1}\right) \\
\Rightarrow \quad \nabla^2 \Phi_{s1} &= 4\pi G\,\rho_{s1}
\end{aligned}$$

To obtain the perturbation on the specific enthalpy, we first note that the pressure can be written in two equivalent ways:

$$\begin{aligned}
p &= p_0 + \varepsilon\,p_1 \\
p(\rho_s) &= p(\rho_0 + \varepsilon\,\rho_{s1}) = p(\rho_0) + \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_{\rho_0} \varepsilon\,\rho_{s1} \\
\Rightarrow \quad p_1 &= \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_{\rho_0} \rho_{s1}
\end{aligned}$$

Similarly,

$$\begin{aligned}
h &= h_0 + \varepsilon\,h_1 \\
h &= h_0 + \left(\frac{\mathrm{d}h}{\mathrm{d}\rho}\right)_{\rho_0} \varepsilon\,\rho_{s1} \\
\Rightarrow \quad h_1 &= \left(\frac{\mathrm{d}h}{\mathrm{d}\rho}\right)_{\rho_0} \rho_{s1} = \left(\frac{\mathrm{d}h}{\mathrm{d}p}\right)_{\rho_0} \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_{\rho_0} \rho_{s1} = \frac{1}{\rho_0} \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_{\rho_0} \rho_{s1} = v_s^2\,\frac{\rho_{s1}}{\rho_0}
\end{aligned}$$

<u>**Problem 3**</u>**:**

Let's consider a sphere with a mass M and with an initial radius h_0, compressed to a radius $h = h_0\,(1-\varepsilon)$, where $0 < \varepsilon << 1$. The density, then, becomes

$$\begin{aligned}
\rho &= \frac{3\,M}{4\pi\,h_0^3}\,\frac{1}{(1-\varepsilon)^3} \simeq \frac{3\,M}{4\pi\,h_0^3}\,\frac{1}{(1-3\varepsilon)} \simeq \rho_0 \cdot (1+3\varepsilon) \\
\Rightarrow \quad \rho_1 &\equiv \rho - \rho_0 \simeq \varepsilon\,\rho_0
\end{aligned}$$

Regarding the pressure p, it is related with the sound speed v_s and we have

$$v_s^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho} \quad ; \quad p = p_0 + p_1 = p_0 + \frac{\mathrm{d}p}{\mathrm{d}\rho}\,\rho_1 = p_0 + v_s^2\,\varepsilon\,\rho_0$$

For the gravitational force, pointing inward:

$$\begin{aligned}
&|F_g|_0 = \frac{GM}{r^2} \qquad \Rightarrow |F_g| = \frac{GM}{(1-\varepsilon)^2 r^2} \approx \frac{GM}{r^2}(1+2\varepsilon) \\
&\Rightarrow |F_g|_1 = |F_g| - |F_g|_0 = \frac{GM}{r^2} 2\varepsilon = \frac{G}{r^2}\frac{8}{3}\pi\varepsilon r^3 \rho_0 \approx G\rho_0\varepsilon r
\end{aligned}$$

The force exerted by the pressure is given by

$$|F_p| = \frac{\nabla p}{\rho}$$

We approximate the gradient by an order-of-magnitude estimate : $\nabla p \approx \frac{p}{r}$ and obtain

$$|F_p|_0 \approx \frac{p}{r\rho} \quad \Rightarrow |F_p| \approx \frac{p_0 + p_1}{r(1-\varepsilon)\rho_0(1+\epsilon)} = \frac{p_0 + p_1}{r\rho_0(1-\epsilon^2)} \approx \frac{p_0 + p_1}{r\rho_0}$$
$$\Rightarrow |F_p|_1 = |F_p| - |F_p|_0 = \frac{p_1}{\rho_0 r} = \frac{v_s^2 \epsilon}{r}$$

The state is in equilibrium if the forces balance each other out. The system is considered stable if the reaction of the system to a contraction is to expand again and revert to its original state, hence when the pressure force is greater than the gravitational force in the case of an initial contraction. This means that the system is unstable if the gravitational force is larger than the pressure force, or

$$|F_g|_1 \gtrsim |F_p|_1 \quad \Rightarrow r^2 \gtrsim \frac{v_s^2}{G\rho_0}$$

Perturbations with a scale larger than $\sim \frac{v_s}{\sqrt{G\rho_0}}$ are unstable.