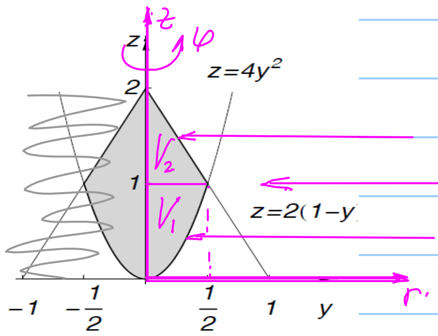
10.4. Exemples additionnels (à discrétion)

1) Volume d'un corps de révolution (exemple)



$z = 2(1-r),\ r \geq 0 \iff r = 1 - \frac{z}{2},\ r \geq 0$

$z = 4r^2,\ r \geq 0 \iff r = \sqrt{\frac{z}{4}},\ r \geq 0$

$\begin{cases} 2\cdot(1-r) = 4r^2 \\ r \geq 0 \end{cases} \Rightarrow r = \frac{1}{2}$

i) Calcul du volume par décomposition du domaine

$$V_1 = \int_0^1 dz \cdot \int_0^{\sqrt{\frac{z}{4}}} dr \int_0^{2\pi} d\varphi \ 1\cdot r$$

$$= 2\pi \int_0^1 dz \left[\tfrac{1}{2} r^2\right]_0^{\sqrt{\frac{z}{4}}} = 2\pi \int_0^1 dz \ \frac{1}{2}\ \frac{z}{4}$$

$$= \frac{\pi}{4}\left[\tfrac{1}{2} z^2\right]_0^1 = \frac{\pi}{8}$$

$$V_2 = \int_1^2 dz \int_0^{1-\frac{z}{2}} dr \int_0^{2\pi} d\varphi \ 1\cdot r$$

$$= 2\pi \int_1^2 dz \left[\tfrac{1}{2} r^2\right]_0^{1-\frac{z}{2}} = 2\pi \int_1^2 dz \ \frac{1}{2}\left(1-\frac{z}{2}\right)^2$$

$$= \pi \left[\left(1-\frac{z}{2}\right)^3 \left(-\frac{2}{3}\right)\right]_1^2 = \frac{2\pi}{3}\left(\frac{1}{2}\right)^3 = \frac{\pi}{12}$$

$$V = V_1 + V_2 = \frac{\pi}{8} + \frac{\pi}{12} = \frac{5\pi}{24}$$

ii) Calcul direct

$$V = \int_0^{\frac{1}{2}} dr \int_{4r^2}^{2(1-r)} dz \int_0^{2\pi} d\varphi \cdot 1 \cdot r$$

$$= 2\pi \cdot \int_0^{\frac{1}{2}} dr \cdot r \cdot \left[z\right]_{4r^2}^{2-2r}$$

$$= 2\pi \int_0^{\frac{1}{2}} dr \left(2r - 2r^2 - 4r^3\right)$$

$$= 2\pi \left[r^2 - \frac{2}{3} r^3 - r^4 \right]_0^{\frac{1}{2}} = \pi \left(\frac{1}{2} - \frac{1}{6} - \frac{1}{8} \right) = \frac{5\pi}{24}$$

iii) Transcription du domaine i) $\longrightarrow$ ii) sans le dessin

V_1 :
$$0 \leqslant z \leqslant 1 \qquad\qquad 0 \leqslant r \leqslant \frac{1}{2}$$
$$\underbrace{0 \leqslant r \leqslant \sqrt{\frac{z}{4}}}_{\Leftrightarrow\ z \geqslant 4r^2} \qquad\qquad 4r^2 \leqslant z \leqslant 1$$

V_2 :
$$1 \leqslant z \leqslant 2 \qquad\qquad 0 \leqslant r \leqslant \frac{1}{2}$$
$$\underbrace{0 \leqslant r \leqslant 1 - \frac{z}{2}}_{\Leftrightarrow\ z \leqslant 2(1-r)} \qquad\qquad 1 \leqslant z \leqslant 2(1-r)$$

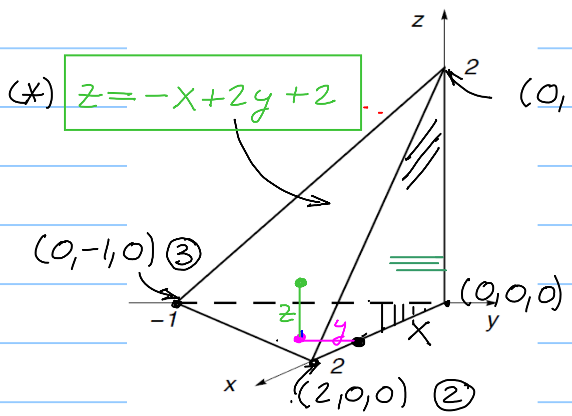
$$V_1 + V_2 = \int_0^{\frac{1}{2}} dr \int_{4r^2}^{1} dz \ \underline{\qquad} + \int_0^{\frac{1}{2}} dr \int_1^{2(1-r)} dz \ \underline{\qquad}$$

$$= \int_0^{\frac{1}{2}} dr \left(\int_{4r^2}^{1} dz \ \underline{\qquad} + \int_1^{2(1-r)} dz \ \underline{\qquad} \right)$$

$$= \int_0^{\frac{1}{2}} dr \int_{4r^2}^{2(1-r)} dz \ \underline{\qquad} = V \quad \text{(méthode ii))}$$

2) Calcul d'un centre de gravité (voir 10.3.3)

(*) $\boxed{z = -x + 2y + 2}$

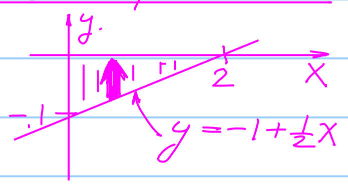


||||| $z = 0$,

/// $y = 0$

$\equiv$ $x = 0$

Dans le plan $z=0$:



(*) :

$$z = ax + by + c =: f(x,y)$$

① $\Rightarrow$ $2 = 0 + 0 + c$ $\Rightarrow$ $c = 2$

② $\Rightarrow$ $0 = a \cdot 2 + 0 + 2$ $\Rightarrow$ $a = -1$

③ $\Rightarrow$ $0 = 0 - b + 2$ $\Rightarrow$ $b = 2$

$$I = \int_0^2 dx \boxed{\int_{-1+\frac{1}{2}x}^{0} dy} \int_0^{-x+2y+2} dz \quad 1$$ (voir les dessins)

aller du plus petit au plus grand !

Même intégrale à partir des inégalités

$$\begin{aligned} 0 &\leq x \\ &y \leq 0 \\ 0 &\leq \underbrace{z \leq -x+2y+2}_{} \end{aligned}$$

$\Leftrightarrow x \leq 2y+2-z$

$\Leftrightarrow y \geq -1+\frac{1}{2}z+\frac{1}{2}x$

$$\Longrightarrow \quad \begin{aligned} &0 \leq x \leq 2 \\ &-1+\tfrac{1}{2}x \leq y \leq 0 \\ &0 \leq z \leq -x+2y+2 \end{aligned}$$

Evaluation de l'intégrale

$$I = \int_0^2 dx \int_{-1+\frac{1}{2}x}^{0} dy \, \underbrace{(-x+2y+2)}_{f(x,y)}$$

$$= \int_0^2 dx \left[-xy+y^2+2y\right]_{y=-1+\frac{1}{2}x}^{y=0}$$

$$= -\int_0^2 dx \left(x\left(1-\tfrac{1}{2}x\right)+\left(1-\tfrac{1}{2}x\right)^2-2+x\right)$$

$$= -\left[\tfrac{1}{2}x^2-\tfrac{1}{6}x^3-\tfrac{2}{3}\left(1-\tfrac{1}{2}x\right)^3-2x+\tfrac{1}{2}x^2\right]_0^2 = \ldots = \frac{2}{3}$$

$$I_1 = \int_0^2 dx \int_{-1+\frac{1}{2}x}^{0} dy \int_0^{-x+2y+2} dz \cdot x = \ldots = \frac{1}{3}$$

$$I_2 = \ldots = -\frac{1}{6}$$

$$I_3 = \ldots = \frac{1}{3}$$

<u>Centre de gravité</u>: $\left(\frac{I_1}{I}, \frac{I_2}{I}, \frac{I_3}{I}\right) = \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}\right)$