Homework 10 Part 1
CS-526 Learning Theory

1. Tensor decomposition

Consider the tensor

$$T = \sum_{i=1}^{K} \lambda_i \, \vec{a}_i \otimes \vec{b}_i \otimes \vec{c}_i$$

where $\vec{a}_i \in \mathbb{R}^{d_1}$ are orthogonal and $\vec{b}_i \in \mathbb{R}^{d_2}$ are orthogonal, $\vec{c}_i \in \mathbb{R}^{d_3}$, and λ_i's are positive and distinct. The goal is to recover the factors $(\lambda_i, \vec{a}_i, \vec{b}_i, \vec{c}_i)$ up to rescaling. Therefore, without loss of generality, we assume that $\|\vec{a}_i\|_2 = \|\vec{b}_i\|_2 = \|\vec{c}_i\|_2 = 1$ for all $1 \leq i \leq K$.

Let $T_{(1)} \in \mathbb{R}^{d_1 \times d_2 d_3}$ be the mode-1 matrization (or unfolding) of T obtained from the vertical fibers of T. $T_{(1)}$ can be expressed in terms of $\lambda_i, \vec{a}_i, \vec{b}_i, \vec{c}_i$'s as:

$$T_{(1)} = \sum_{i=1}^{K} \lambda_i \, \vec{a}_i \big(\vec{c}_i \otimes_{\text{Kro}} \vec{b}_i\big)^{\text{T}}$$

with $\otimes_{\text{Kro}}$ denoting the Kronecker product of two vectors:

$$x \otimes_{\text{Kro}} y = [x_1 y^{\text{T}}, x_2 y^{\text{T}}, \cdots, x_n y^{\text{T}}]^{\text{T}} \in \mathbb{R}^{nm} \qquad \text{for } x \in \mathbb{R}^n, y \in \mathbb{R}^m$$

1. Let $X = T_{(1)} T_{(1)}^{\text{T}}$. Express X in terms of $\lambda_i, \vec{a}_i, \vec{b}_i, \vec{c}_i$'s, and write its spectral decomposition. What is the rank of X? Explain how to recover the vectors $\vec{a}_i$'s and corresponding λ_i's.
2. Explain how to recover the vectors $\vec{b}_i$'s and how to pair them with the $\vec{a}_i$'s and λ_i's.
3. Now that we have found $(\lambda_i, \vec{a}_i, \vec{b}_i)$'s, describe a way to recover $\vec{c}_i$'s.

   Hint: Try multilinear transformations of T!

2. First exercise of Hmw 10-2