Solutions Homework 9
CS-526 Learning Theory

Problem 1: Jennrich's type algorithm for order 4 tensors

1) To apply Jennrich's algorithm we need to prove that the matrix $E = [\underline{c}_1 \otimes_{\text{Kro}} \underline{d}_1, \dots, \underline{c}_R \otimes_{\text{Kro}} \underline{d}_R]$ is full column rank (A, B are full column rank by assumption). Note that the same proof as the one in Problem 4 question 1 applies. Nevertheless we repeat the argument here.
Let $\underline{v} \in \mathbb{R}^R$ a column vector in the kernel of E, i.e., $E\underline{v} = 0$. Then:

$$\forall \gamma \in [I_3] : \sum_{r=1}^{R} (c_r^\gamma v^r) \underline{d}_r = 0 \Rightarrow \forall \gamma \in [I_3], \forall r \in [R] : c_r^\gamma v^r = 0 \Rightarrow C\underline{v} = 0 \Rightarrow \underline{v} = 0 .$$

The first implication follows from D being full column rank and the third one from C being full column rank. We conclude that the kernel of E is $\{0\}$: E is full column rank.
We can therefore apply Jennrich's algorithm.

2) We recover the rank R as well as A, B and E by applying Jennrich's algorithm to $\widetilde{T}$. From E we can then determine C and D. Fix $r \in [R]$. Since C is full column rank, there exists $\alpha_r \in [I_3]$ such that $c_r^{\alpha_r} \neq 0$. As $c_r^{\alpha_r} \neq 0$, the I_4-dimensional column vector $\underline{\tilde{d}}_r = c_r^\alpha \underline{d}_r$ contained in the r^{th} column of E recovers $\underline{d}_r$ up to some feature scaling. Doing this for every $r \in [R]$ we build the matrix $\widetilde{D} = \begin{bmatrix} \underline{\tilde{d}}_1 & \underline{\tilde{d}}_2 & \dots & \underline{\tilde{d}}_R \end{bmatrix}$ that recovers D up to some feature scaling and is full column rank (because D is). Finally, for every $r \in R$, pick $\beta_r \in [I_4]$ such that $\tilde{d}_r^{\beta_r} \neq 0$ (such β_r exists because $\widetilde{D}$ is full column rank) and use the entries of E corresponding to $c_r^\alpha d_r^{\beta_r}$, $\alpha \in [I_3]$, to build the vector $\underline{\tilde{c}}_r = \frac{d_r^{\beta_r}}{\tilde{d}_r^{\beta_r}} \underline{c}_r$. The matrix $\widetilde{C} = \begin{bmatrix} \underline{\tilde{c}}_1 & \underline{\tilde{c}}_2 & \dots & \underline{\tilde{c}}_R \end{bmatrix}$ recovers C up to some feature scaling.

Problem 2: short questions and answers

B is true. If $w_i(\epsilon) \neq 0$ for all i then the three arrays have rank K and there are K terms in the tensor decomposition. Therefore, by Jennrich's theorem, the decomposition is unique and the rank of the tensor $T(\epsilon)$ is K.

A is not true if there exists some (i, ϵ) such that $w_i(\epsilon)$ is zero.

C is not true because all the functions w_i are continuous. Therefore, $\lim_{\epsilon \to 0} T(\epsilon) = T(0)$ and, by Jennrich's theorem, the rank is at most K (the rank is K if $\forall i : w_i(0) \neq 0$).

D is not true because if $w_i(\epsilon) \neq 0$ for all i and $\epsilon \in [0, 1]$ then the rank is K.