Astrophysics IV, Dr. Yves Revaz | EPFL

4th year physics | Exercises week 10

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Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

a) Spherical Cordinates:

Inserting the gradient expression into the Boltzmann equation in Cartesian coordinates and writing the scalar products explicitly gives:

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r} \frac{\partial f}{\partial \theta} + \frac{v_\phi}{r \sin \phi} \frac{\partial f}{\partial \phi} + \dot{v}_r \frac{\partial f}{\partial v_r} + \dot{v}_\theta \frac{\partial f}{\partial v_\theta} + \dot{v}_\phi \frac{\partial f}{\partial v_\phi} = 0 \tag{1}$$

We have

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta}, \quad v_\phi = r \sin\theta \dot{\phi}$$

and the Lagrangian

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2\right) - \Phi(r)$$

To obtain the Equations of Motion, we use

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For $q_i = r$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{r} - r\dot{\theta}^2 - r \sin^2\theta \dot{\phi}^2 + \frac{\partial \Phi}{\partial r} = 0$$
$$\dot{v}_r = \frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial \Phi}{\partial r}$$

For $q_i = \theta$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) - r^2 \sin\theta \cos\theta \ \dot{\phi}^2 = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(r \cdot (r\dot{\theta})\right) - (r \sin\theta \ \dot{\phi})^2 \frac{\cos\theta}{\sin\theta}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\theta}\right) = \frac{1}{r}\left(\frac{(r\sin\theta \ \dot{\phi})^2}{\tan\theta} - \dot{r}(r\dot{\theta})\right)$$
$$\dot{v}_\theta = \frac{1}{r}\left(\frac{v_\phi^2}{\tan\theta} - v_r v_\theta\right)$$

For $q_i = \phi$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2 \sin^2\theta\ \dot{\phi}\right) &= 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(r\sin\theta \cdot (r\sin\theta\dot{\phi})\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(r\sin\theta\ \dot{\phi}\right) &= -(r\sin\theta\ \dot{\phi})\left(\frac{\dot{r}}{r} + \frac{r\dot{\theta}}{r\tan\theta}\right) \\
\dot{v}_\phi &= -\frac{v_\phi}{r}\left(v_r + \frac{v_\theta}{\tan\theta}\right)
\end{aligned}$$

Inserting $\dot{v}_r$, $\dot{v}_\theta$, $\dot{v}_\phi$ into expression (1) gives the required result.

b) Cylindrical Cordinates:

Inserting the gradient expression into the Boltzmann equation in Cartesian coordinates and writing the scalar products explicitly gives:

$$\frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\phi}{r}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z} + \dot{v}_r\frac{\partial f}{\partial v_r} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z} = 0 \tag{2}$$

We have

$$v_r = \dot{r}, \quad v_\phi = r\dot{\phi}, \quad v_z = \dot{z}$$

and the Lagrangian

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \Phi(r, z)$$

To obtain the Equations of Motion, we use

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For $q_i = r$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\dot{r} - r\dot{\phi}^2 + \frac{\partial \Phi}{\partial r} &= 0 \\
\dot{v}_r &= \frac{v_\phi^2}{r} - \frac{\partial \Phi}{\partial r}
\end{aligned}$$

For $q_i = \phi$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\phi}\right) - 0 &= 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(r \cdot (r\dot{\phi})\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\phi}\right) &= -\frac{1}{r}\dot{r} \cdot r\dot{\phi} \\
\dot{v}_\phi &= -\frac{v_r v_\phi}{r}
\end{aligned}$$

For $q_i = z$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{z} + \frac{\partial \Phi}{\partial z} = 0$$
$$\dot{v}_z = -\frac{\partial \Phi}{\partial z}$$

Inserting $\dot{v}_r$, $\dot{v}_\phi$, $\dot{v}_z$ into expression (2) gives the required result.

Problem 2:

In Cartesian coordinates, the collisionless Boltzmann equation writes:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{3}$$

The only term that will be influenced by the rotation is the acceleration $\mathbf{a}$. In a rotating frame, the Lagrangian is:

$$L = \frac{1}{2}\left(\mathbf{v} + \mathbf{\Omega} \times \mathbf{x}\right)^2 - \Phi(\mathbf{x}) \tag{4}$$

Defining the effective potential as:

$$\Phi_{\text{eff}}(\mathbf{x}) = \Phi(\mathbf{x}) - \frac{1}{2}\left(\mathbf{\Omega} \times \mathbf{x}\right)^2, \tag{5}$$

the Lagrangian writes:

$$L = \frac{1}{2}\mathbf{v}^2 + \mathbf{v}\left(\mathbf{\Omega} \times \mathbf{x}\right) - \Phi_{\text{eff}}(\mathbf{x}). \tag{6}$$

Using the Euler-Lagrange equation we get:

$$\mathbf{a} = -\nabla\Phi_{\text{eff}}(\mathbf{x}) - 2\left(\mathbf{\Omega} \times \mathbf{v}\right) \tag{7}$$

We conclude that the collisionless Boltzmann equation in the rotating frame writes:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \left[\nabla\Phi_{\text{eff}}(\mathbf{x}) + 2\left(\mathbf{\Omega} \times \mathbf{v}\right)\right] \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{8}$$