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# Spring 2024: Final Project

## COM-308: Quantum Computing

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### Notations

We denote  $|i\rangle$  the state in the computational basis corresponding to the binary decomposition of  $i$ . For example, with 4 qubits,  $|5\rangle = |0101\rangle$ .

### The HHL Algorithm

The HHL (Harrow-Hassidim-Lloyd) algorithm is a quantum algorithm designed to solve systems of linear equations. Given a matrix  $A \in \mathbb{C}^{2^n \times 2^n}$  and a vector  $\mathbf{b} \in \mathbb{C}^{2^n}$ , we want to find a vector  $\mathbf{x} \in \mathbb{C}^{2^n}$  that verifies

$$A\mathbf{x} = \mathbf{b}. \quad (1)$$

The HHL algorithm is exponentially faster than classical methods when the matrix  $A$  is sparse. If  $s$  is the number of non-zero elements per row in  $A$ , then the quantum circuit can be constructed with  $O(ns^2)$  gates whereas the best classical algorithm runs in  $O(2^n s)$ .

To solve the linear system  $A\mathbf{x} = \mathbf{b}$  with a quantum circuit, we need to represent  $\mathbf{b}$  and  $\mathbf{x}$  by quantum states; thus, we need to scale them to unit length  $\|\mathbf{b}\| = \|\mathbf{x}\| = 1$ . Then  $\mathbf{b}$  can be represented by a state  $|b\rangle$  using  $n$  qubit such that  $|b\rangle = \sum_{i=0}^{2^n-1} b_i |i\rangle$ . Here, the  $b_i$  are the components of  $\mathbf{b}$ . The vector solution  $\mathbf{x}$  can be then represented by the state  $|x\rangle$  that verifies

$$|x\rangle = cA^{-1}|b\rangle, \quad c^{-1} = \|A^{-1}|b\rangle\| \quad (2)$$

where  $c$  ensures that the state is normalized.

The HHL algorithm uses quantum phase estimation to encode the solution  $\mathbf{x}$  into a quantum state. To do so, the algorithm requires that the matrix  $A$  be Hermitian.

**Question Theory 1:** Show that if  $A$  is not Hermitian, we can still find a solution to the system by running the HHL algorithm on the larger system:

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}. \quad (3)$$

So, from now on, we will assume that  $A$  is a Hermitian matrix, i.e.  $A = A^\dagger$ . Thus, by the spectral theorem (see Homework 1 Ex 3), there exists a set of orthogonal states  $(|u_i\rangle)_{i=0,\dots,2^n-1}$  such that  $A$  can be written

$$A = \sum_{i=0}^{2^n-1} \lambda_i |u_i\rangle\langle u_i| \quad (4)$$

where the  $\lambda_i \in \mathbb{R}$  are the eigenvalues of  $A$ . The  $(|u_i\rangle)_{i=0,\dots,2^n-1}$  form an eigenbasis of  $A$ . The state  $|b\rangle$  can also be written in the  $(|u_i\rangle)_{i=0,\dots,2^n-1}$  basis and we denote

$$|b\rangle = \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle. \quad (5)$$

**Question Theory 2:** Check that  $|x\rangle = c \sum_{i=0}^{2^n-1} \frac{\beta_i}{\lambda_i} |u_i\rangle$  is solution to the system.

## The circuit

The HHL circuit is represented in Figure 1. The circuit uses three registers:

- The top register is 1 ancilla qubit initialized to  $|0\rangle$ .
- The middle register is a memory register that stores the eigenvalues  $\lambda_i$  of  $A$ . More precisely, we will store the binary representation of  $\lambda_i$ . The number of qubits  $m$  needed for this register will therefore depend on  $\lambda_i$ . This register is initialized to  $|0\rangle^{\otimes m}$ .
- The bottom register uses  $n$  qubits and is initialized with the state  $|b\rangle$ .

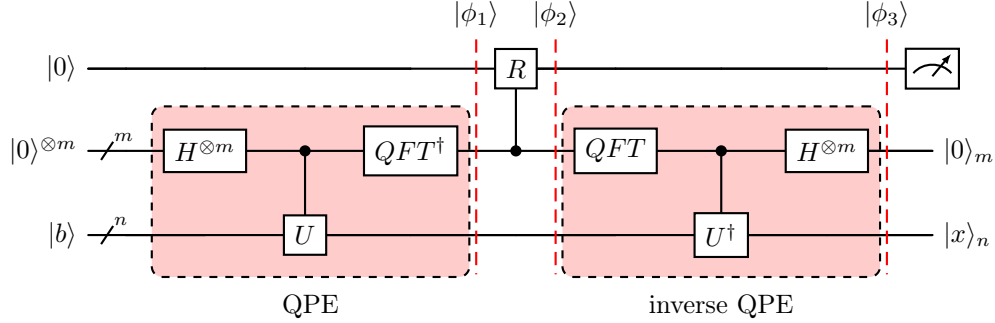


Figure 1: Quantum Circuit for HHL Algorithm

The circuit is composed of 4 steps, a quantum phase estimation (QPE), a controlled rotation, an inverse quantum phase estimation and a measurement. Let us detail the gates appearing in each part:

**Quantum phase estimation:** This part of the circuit is detailed in Figure 2. The circuit starts with a Hadamard gate on each qubit of the memory register. The unitary  $U$  is

$$U = e^{i2\pi \frac{A}{2^m}}. \quad (6)$$

Then we apply an inverse quantum Fourier transform on the memory register.

$$QFT^\dagger |k\rangle = \frac{1}{\sqrt{2^m}} \sum_{j=0}^{2^m-1} e^{-i2\pi \frac{kj}{2^m}} |j\rangle. \quad (7)$$

**Controlled rotation:** The gate  $R$  realizes the transformation

$$R(|0\rangle \otimes |\lambda\rangle) = \left( \sqrt{1 - \frac{1}{\lambda^2}} |0\rangle + \frac{1}{\lambda} |1\rangle \right) \otimes |\lambda\rangle. \quad (8)$$

**Inverse QPE:** We apply the **inverse** gates of the QPE in **reverse** order to set back the memory register to  $|0\rangle^{\otimes m}$ . The memory register is no longer entangled with the output register.

**Measurement:** The algorithm outputs  $|x\rangle$  if the ancilla qubit is measured in state  $|1\rangle$ .

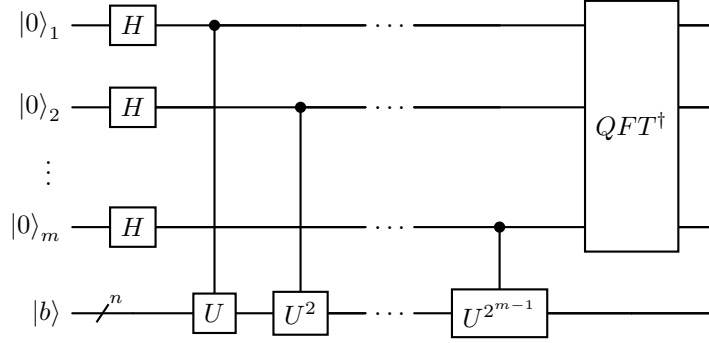


Figure 2: Detailed QPE

## Analysis

We assume that the eigenvalues of  $A$  are positive integers, i.e.  $\lambda_i \in \mathbb{N}^*$  and  $\max_i(\lambda_i) < 2^m$ . The circuit starts with a quantum phase estimation that stores the eigenvalues in the memory register.

**Question Theory 3:** Show that the state  $|\phi_1\rangle$  defined in Figure 1 is

$$|\phi_1\rangle = |0\rangle \otimes \left( \sum_i \beta_i |\lambda_i\rangle \otimes |u_i\rangle \right). \quad (9)$$

*Hint: Use the lecture notes on QPE. You can start by answering these questions:*

- What are the eigenvalues and eigenvectors of  $U$  ?
- What happens if  $|b\rangle = |u_i\rangle$  ?

Then we apply a controlled rotation  $R$  to create the  $\frac{1}{\lambda}$  factor.

**Question Theory 4:** Compute  $|\phi_2\rangle$ .

We want to disentangle the memory register from the output state. Thus, we apply the inverse QPE.

**Question Theory 5:** Show that the state  $|\phi_3\rangle$  defined in Figure 1 is

$$|\phi_3\rangle = \sum_i \beta_i \left( \sqrt{1 - \frac{1}{\lambda_i^2}} |0\rangle + \frac{1}{\lambda_i} |1\rangle \right) \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle. \quad (10)$$

*Hint: Start with  $|b\rangle = |u_i\rangle$  and use the fact that gates are unitary.*

**Question Theory 6:** Show that the output of the circuit is a solution of the linear system if the result of the measurement is "1". What is the probability of obtaining this result? Use  $\max_i(\lambda_i) < 2^m$  to lower bound this result.

**Question Implementation 1:** Implement the HHL circuit on a simulator. Consider only the case where the eigenvalues of  $A$  are powers of 2. More detailed instructions are given in the notebook.

## Measuring the solution

We want to measure  $|x\rangle$  to learn its value. We are used to measuring states with projectors. The probability of a state  $|\phi\rangle$  to be in state  $|\psi\rangle$  is

$$P(|\psi\rangle) = \langle\phi|(|\psi\rangle\langle\psi|)|\phi\rangle = |\langle\phi|\psi\rangle|^2. \quad (11)$$

**Question Implementation 2:** On a simulator and a real device, run the HHL circuit to measure  $|x\rangle$ , the solution of the system

$$A = \frac{1}{2} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } |b\rangle = \frac{1}{\sqrt{5}}(2|0\rangle - |1\rangle). \quad (12)$$

Compare it to the expected solution  $\frac{A^{-1}\mathbf{b}}{\|A^{-1}\mathbf{b}\|}$ . What is missing to fully reconstruct the solution?

## Limitations

Although the HHL algorithm offers exponential speedup over classical methods, it has several limitations:

- It requires that the matrix  $A$  be sparse to reach the announced complexity. In our implementation, we do not pay attention to this requirement.
- We need to have some a priori knowledge of the eigenvalues  $\lambda_i$  to have an exact and efficient algorithm.
- The algorithm only gives a solution with some probability of error.
- The preparation of the quantum state for  $|b\rangle$  is often difficult and requires additional resources.
- We do not have direct access to the result. We still need to measure  $|x\rangle$  to learn the solution.
- On a real quantum noisy device, we will only have access to an approximate solution.

## Bonus - Another measurement

If we measure our state  $|x\rangle$  only in the computational basis, which means that we choose the set of projectors  $|i\rangle\langle i|$ , we will only obtain the norm of each amplitude in the computational basis. If the amplitude is complex, we will not be able to reconstruct the full state  $|x\rangle$ . Thus, we need another kind of measurement.

An observable  $O$  is a Hermitian matrix that represents a physical property we want to measure in our system. It can be the position, momentum, spin ... We are interested to know the average value of that property. As a Hermitian matrix,  $O$  can be decomposed into its eigenbasis,  $O = \sum_k \mu_k |v_k\rangle\langle v_k|$ . The expected value of  $O$  observed in the state  $|\phi\rangle$  is then  $\langle O \rangle = \langle \phi | O | \phi \rangle = \sum_k \mu_k \langle \phi | (|v_k\rangle\langle v_k|) | \phi \rangle = \sum_k \mu_k P(|v_k\rangle)$ .

Remark that if we choose  $O$  to be a projector  $|\psi\rangle\langle\psi|$ , then  $O$  has only one non-zero eigenvalue which is 1 associated with the eigenvector  $|\psi\rangle$ . Therefore, the expected value of  $O$  is  $P(|\psi\rangle)$ .

We focus on the case where  $A$  is a  $2 \times 2$  matrix and  $\mathbf{b}$  is a vector of size 2. We denote by  $\rho$  the matrix  $\rho = |x\rangle\langle x|$ . The set of operators  $P = \frac{1}{\sqrt{2}}\{I, X, Y, Z\}$  is a basis of the space of the  $2 \times 2$  matrices. Thus, we can decompose  $\rho$  in this basis:

$$\rho = \frac{c_I I + c_X X + c_Y Y + c_Z Z}{\sqrt{2}} \quad (13)$$

**Question Theory 7:** Show that  $c_\sigma = \text{Tr}(\rho\sigma)$  for all  $\sigma \in P$ .

*Hint: Show that for all  $\sigma_i \sigma_j \in P$ ,  $\text{Tr}(\sigma_i \sigma_j) = 0$  if  $j \neq i$  and 1 if  $i = j$ .*

Pauli matrices are Hermitian and can be interpreted as observables. Therefore, by measuring each operator  $\sigma \in P$ , we can have access to their expected value  $\langle \sigma \rangle = \langle x | \sigma | x \rangle = \text{Tr}(\langle x | \sigma | x \rangle) = \text{Tr}(\rho\sigma)$ , and we can reconstruct the density matrix  $\rho$

$$\rho = \frac{\langle I \rangle I + \langle X \rangle X + \langle Y \rangle Y + \langle Z \rangle Z}{2}. \quad (14)$$

**Question Implementation 3:** Run your HHL circuit on a simulator and a real quantum device and reconstruct completely the solution state for the system defined as

$$A = \frac{1}{9} \begin{pmatrix} 13 & 2 + i4 \\ 2 - i4 & 14 \end{pmatrix} \text{ and } |b\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \quad (15)$$

Observable measurements will be detailed in the notebook.