Equilibria of collisionless systems

2rd part

Outlines

The Jeans theorems

- Steady-state solutions of the Collisionless Boltzmann equation
- Symmetry and integrals of motion

Connections between DFs and orbits

Connections between barotropic fluids and ergodic stellar systems

Self-consitent spherical models with Ergodic DF

- DFs from mass distribution
    - The Eddington formula
    - Examples
- Models defined from DFs
    - Polytropes and Plummer models

Equilibria of collisionless systems

The Jeans Theorems

Question :

$\boxed{\text{How can we obtain a steady-state solution of the collision-less Boltzmann equation? } \quad \frac{\partial f}{\partial t} = 0}$

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \underbrace{\frac{\partial H}{\partial p}}_{\dot{q}} \frac{\partial f}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p} = 0$$

In carthesian coordinates $\quad \frac{\partial H}{\partial \bar{x}} = \frac{\partial \phi}{\partial \bar{x}}$

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \frac{\partial f}{\partial \bar{x}} v - \frac{\partial \phi}{\partial \bar{x}} \frac{\partial f}{\partial \bar{v}} = 0$$

Back to the integrals of motion

The function $I(\vec{x}(t), \vec{v}(t))$ is an integral of motion if

$$\boxed{\frac{d}{dt} I(\vec{x}(t), \vec{v}(t)) = 0}$$ along the trajectory.

But

$$\frac{dI}{dt} = \frac{\partial I}{\partial \vec{x}} \dot{\vec{x}} + \frac{\partial I}{\partial \vec{v}} \dot{\vec{v}} = 0$$

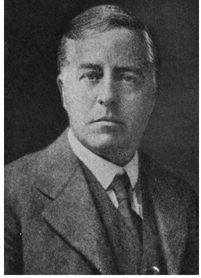
$$= \boxed{\frac{\partial I}{\partial \vec{x}} \vec{v} - \frac{\partial I}{\partial \vec{v}} \vec{\nabla} \phi = 0}$$

Similar to the Collisionless Boltzmann equation

If $I(\vec{x}, \vec{v})$ is an integral of motion

$I(\vec{x}, \vec{v})$ is a steady state solution of the Collisionless Boltzbann equation

James Jeans
1877-1947



Jeans theorems

I. Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion.

II. Any function of integrals of motion yields a steady-state solution of the collisonless Boltzmann equation.

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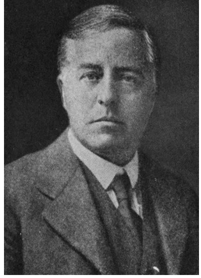

Jeans theorems

I. Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion.

Demonstration:

If a function is a solution of the steady-state collisionless Boltzmann equation, then, it is an integral of motion, thus the function depends on the phase-space coordinates only through integrals of motion (itself !).

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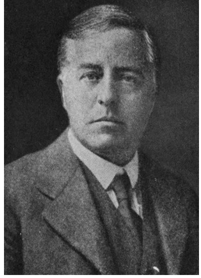

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Demonstration:

Assume $$f(\vec{x},\vec{v}) = f(I_1(\vec{x},\vec{v}), I_2(\vec{x},\vec{v}), I_3(\vec{x},\vec{v}), ...)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{x},\vec{v}) = \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_1}{\mathrm{d}t}}_{=0} + \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_2}{\mathrm{d}t}}_{=0} + \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_3}{\mathrm{d}t}}_{=0} + ... = 0$$

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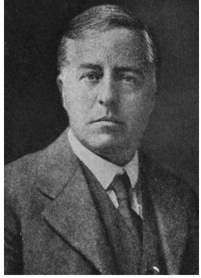

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Extremely useful to generate DFs

Demonstration:

Assume $$f(\vec{x},\vec{v}) = f(I_1(\vec{x},\vec{v}), I_2(\vec{x},\vec{v}), I_3(\vec{x},\vec{v}), ...)$$

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Equilibria of collisionless systems

Symmetries and DFs

Choices of DFs and relations with the velocity moments

1. DFs that depend only on H

(no particular symmetry except time!)

$$\phi = \phi(\vec{x}, \cancel{t})$$

Ergodic distribution functions

Example

$$\begin{cases} H(\vec{x}, \vec{v}) = \frac{1}{2}\vec{v}^{\,2} + \phi(\vec{x}) \\[2ex] f = f\left(\frac{1}{2}\vec{v}^{\,2} + \phi(\vec{x})\right) \end{cases}$$

Note: the velocity dependency is only through v^2 (isothropic)

Mean velocity

$$\bar{\vec{v}}(\vec{x}) = \frac{1}{\nu(\vec{x})} \int \vec{v}\, f\left(\frac{1}{2}\vec{v}^{\,2} + \phi(\vec{x})\right) d^3\vec{v} = 0$$

indeed

$$\bar{v}_x(\vec{x}) = \frac{1}{\nu(\vec{x})} \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_x\, \underbrace{v_x}_{\text{odd}}\, \underbrace{f\left(\frac{1}{2}\vec{v}^{\,2} + \phi(\vec{x})\right)}_{\text{even}} = 0$$

1. DFs that depend only on $\mathcal{H}$

Velocity dispersions

$$\sigma_{ij}^2 = \frac{1}{\nu(\vec{x})} \int \underbrace{(v_i - \cancel{\bar{v}_i})(v_j - \cancel{\bar{v}_j})}_{\text{odd, except if } i=j} \, \delta\left(\tfrac{1}{2}\vec{v}^2 + \phi(\vec{x})\right) d^3\vec{v}$$

($\cancel{\bar{v}_i} = 0$, $\cancel{\bar{v}_j} = 0$)

$$= \delta_{ij}\sigma^2$$

odd, except if $i=j$ $\quad (\sigma_{xx} = \sigma_{yy} = \sigma_{zz})$

$$\sigma^2 = \frac{1}{\nu(\vec{x})} \int_{-\infty}^{\infty} v_z^2 \, dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_x \, \delta\left(\tfrac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$$

using spherical coord in velocity space :
$$\left\{ \begin{array}{l} dv_x\, dv_y\, dv_z = v^2 \sin\theta \, dv\, d\varphi\, d\theta \\ v_z^2 = v^2 \cos^2\theta \\ v^2 = v_x^2 + v_y^2 + v_z^2 \end{array} \right.$$

$$\boxed{\sigma^2 = 4\pi \frac{1}{\nu(x)} \int_0^{\infty} v^4 \, \delta\left(\tfrac{1}{2}v^2 + \phi(\vec{x})\right) dv}$$

$$\sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

isothropic system :

the velocity ellipsoid is a sphere

Note: The term "ergodic" denotes a system that uniformly explores its energy surface in phase space:

$$\boxed{\begin{array}{l}\Rightarrow \text{ the distribution function is} \\ \quad\;\; \text{uniform on the energy surface} \\[1em] \qquad\qquad\qquad \varrho = \varrho(E)\end{array}}$$

2. DFs that depend on H and $\vec{L}$

(spherical symmetry)

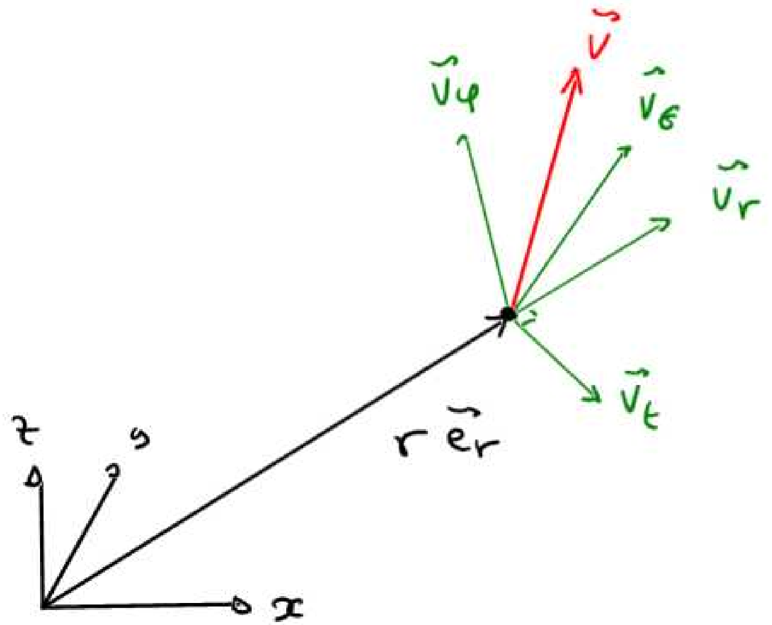
$$\phi = \phi(r)$$

We restrict our study to symmetric DFs : indep. of any direction

$$f(\vec{x}, \vec{v}) = f(H, |L|) \qquad \vec{L} \to |\vec{L}|$$

We consider the system in spherical coordinates

$$r \quad \theta \quad \varphi \qquad \vec{v}_r \quad \vec{v}_\theta \quad v_\varphi$$

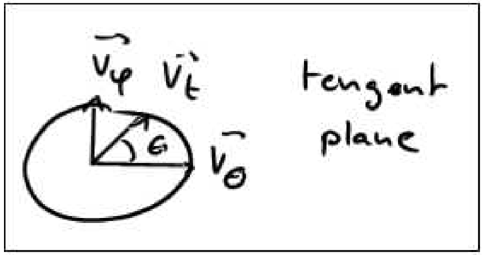


$\vec{v}_r$: radial velocity $\quad \vec{V}_r = \left(\vec{V} \cdot \vec{e}_r\right) \vec{e}_r$

$\vec{v}_t$: tangential velocity $\quad \vec{V}_t = |\vec{V} \times \vec{e}_r| \vec{e}_t$

$$\vec{v} = \vec{v}_r + \vec{v}_t$$

$$= \vec{v}_r + \vec{v}_\theta + \vec{v}_\varphi$$



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Mean velocities

$$\bar{v}_r = 0 \qquad \bar{v}_t = 0$$

EXERCICE

Velocity dispersions

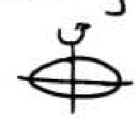
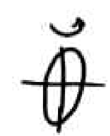
$$\sigma_r^2 \neq 0$$

$$\sigma_\theta^2 = \sigma_\varphi^2 \neq 0$$

Anisothropic system

$$\boxed{\sigma_r^2 \neq \sigma_\theta^2 = \sigma_\varphi^2}$$

The velocity ellipsoid is

oblate  or prolate 

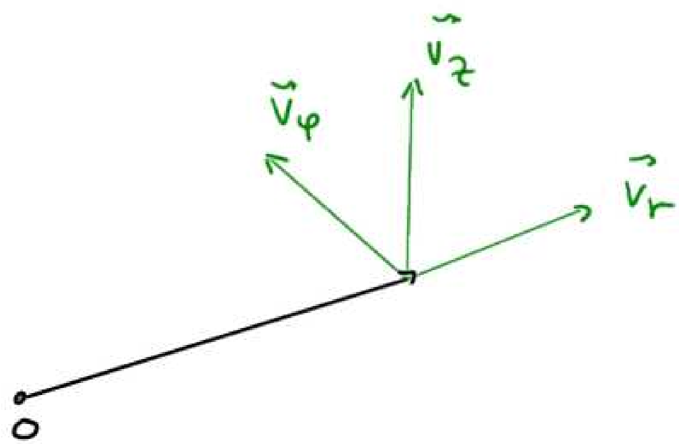
$\boxed{\text{3. DFs that depend on } H \text{ and } L_z}$

(cylindrical symmetry)

$$\phi = \phi(R, |z|)$$

$$f(\vec{x}, \vec{v}) = f(H, L_z)$$

We consider the system in cylindrical coordinates



$$r \quad \varphi \quad z \qquad \vec{v}_r \quad \vec{v}_\varphi \quad \vec{v}_z$$

3. DFs that depend on H and L_z

(cylindrical symmetry)

$$\phi = \phi(R, |z|)$$

$$f(\vec{x}, \vec{v}) = f(H, L_z)$$

Mean velocities

$$\bar{v}_R = 0 \qquad \bar{v}_z = 0 \qquad \bar{v}_\varphi \neq 0$$

Velocity dispertions

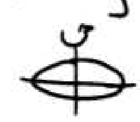
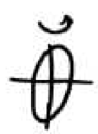
EXERCICE

$$\sigma_\varphi^2 \neq 0$$

$$\sigma_R^2 = \sigma_z^2 \neq 0$$

Anisothropic system

$$\boxed{\sigma_\varphi^2 \neq \sigma_R^2 = \sigma_z^2}$$

The velocity ellipsoid is oblate  or prolate 

Equilibria of collisionless systems

Connections between DFs and orbits

Example 1 — 1-D potential

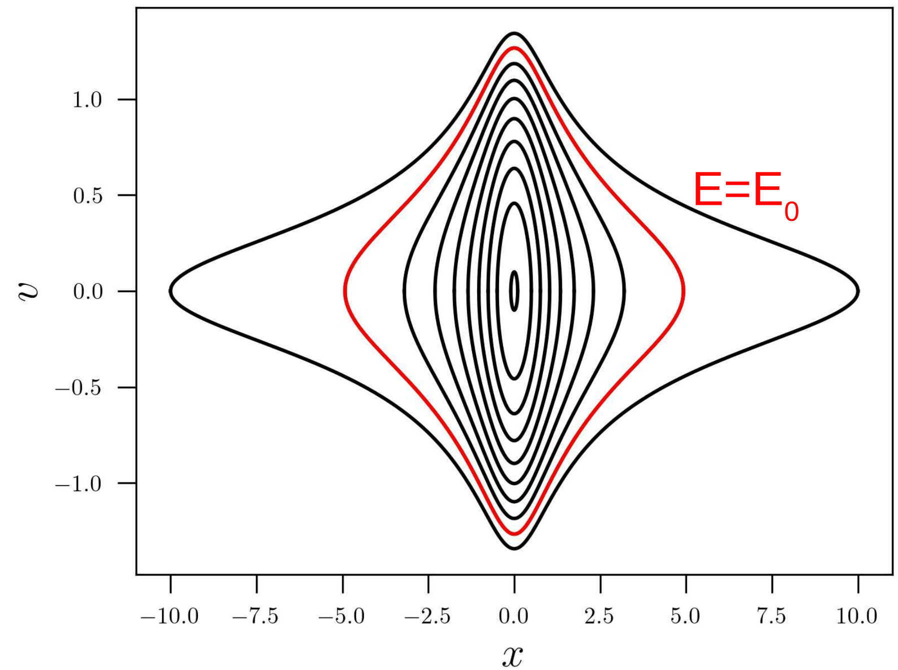
$$\begin{cases} E = \frac{1}{2} v^2 + \phi(x) \\ v = \pm \sqrt{2(E - \phi(x))} \end{cases}$$

a) $f(x,v) = f(E) = \delta(E - E_0)$

$$\begin{cases} \infty & v = \pm \sqrt{2(E_0 - \phi(x))} \\ 0 & \text{instead} \end{cases}$$

b) $f(x,v) = f(E)$

↳ give a weight to orbits depending on their energy



Example 2

3D sperical potential

→ planar orbits described by $E, |\vec{L}|$

a) **Ergodic DF** : $f(\vec{x},\vec{v}) = f(E(\vec{x},\vec{v}))$

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

- model built-out of all orbits of all planes with a weight that depend on their energy

radial and circular orbits have the same weight

b) **Non-ergodic DF** : $f(\vec{x},\vec{v}) = f(E(\vec{x},\vec{v}), |\vec{L}|(\vec{x},\vec{v}))$

$$\boxed{\sigma_r^2 \neq \sigma_\theta^2 = \sigma_\varphi^2}$$

- model built-out of all orbits of all planes with a weight that depend only on their energy and angular momentum

radial and circular orbits are weighted differently

c) <u>Non-ergodic DF</u>: $f(\vec{x},\vec{v}) = f(E(\vec{x},\vec{v}), \vec{L}) = f_E(E)\, f_L(\vec{L})$

! not spherical

$$f_L(\vec{L}) \begin{cases} \neq 0 & \text{if } \vec{L} \,/\!/\, \vec{e}_z \\ = 0 & \text{instead} \end{cases}$$

$$\boxed{\sigma_\varphi^2 \neq \sigma_R^2 = \sigma_z^2}$$

- model built-out of orbits in the $z=0$ plane with a weight that depend only on their energy and angular momentum

Questions

Why an ergodic DF <u>with a priori no constraint on the symmetry of the potential</u> leads to an <u>isotropic</u> velocity dispersion tensor ?

$$\Phi(r) \qquad f(H) \qquad \Longrightarrow \qquad \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

Connections between barotropic fluids and ergodic stellar systems

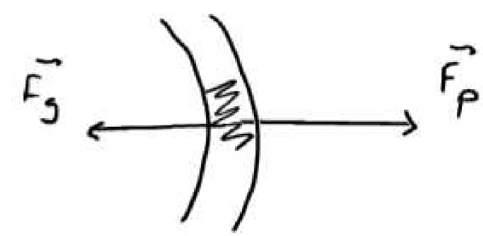
Connections between fluids and stellar systems

In fluid dynamics, the properties of a fluid at rest in a potential is obtained through the Euler equation

$$\boxed{\frac{d\vec{v}}{dt} = \underbrace{-\frac{\vec{\nabla} P}{\rho}}_{\text{pressure force}} \underbrace{- \vec{\nabla}\phi}_{\text{gravity}}}$$

<u>At rest</u> $\quad \frac{d\vec{v}}{dt} = 0$

$$\boxed{\frac{\vec{\nabla} P}{\rho} = - \vec{\nabla}\phi}$$



<u>In 1-D</u> (isothropic case)

$$\boxed{\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\partial \phi}{\partial r}}$$

Equation of state (EoS) $\quad P = P(\rho, T)$

$P = P(\rho)$: barotropic (depends only on the density)

$P = K\rho^{\gamma}$: polytropic

$P = \frac{k_B T}{m}\rho$: isotherm ($T = \text{cte}$)

Together with the hydrostatic equation,

$$\boxed{\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\partial \phi}{\partial r}}$$

this relates $\underline{\rho(r) \text{ with } \phi(r)}$.

Self - gravity

The Poisson equation

$$\boxed{\vec{\nabla}^2 \phi = 4\pi G \rho}$$

This constraints the potential $\phi(r)$

or equivalently the density $\rho(r)$

Indeed:

$$\boxed{\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\partial \phi}{\partial r}} \quad + \quad P(\rho) \quad + \quad \boxed{\vec{\nabla}^2 \phi = 4\pi G \rho}$$

$\Rightarrow$ diff. equation for $\phi(r)$ or $\rho(r)$

Note An ergodic DF is such that the velocity dispertion is isothropic

$(\sigma_\sigma \ {}_\sigma)$ $\equiv$ similar to a gaseous system

Idea : define a function $P(\rho)$ (an equivalent of the pressure) which is such that :

$$\boxed{\frac{\vec{\nabla} P}{\rho} = - \vec{\nabla} \phi} \qquad \frac{1}{\rho}\frac{\partial P}{\partial r} = - \frac{\partial \phi}{\partial r}$$

if spherical

If we find $P(\rho)$ for our stellar system, its density will be the same than the one of a gaseous system as the "pressure" will be equivalent.

<u>Ergodic DF</u> $\quad f(\vec{x},\vec{v}) = f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$

<u>Density</u> $\quad \rho(\vec{x}) = \int d^3v \; f(\vec{x},\vec{v})$

$$= \int d^3v \; f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$$

as ρ depends on $\vec{x}$ only through ϕ, we can write:

$\rho = \rho(\phi)$ and assuming it to be bijective

$$\boxed{\phi = \phi(\rho)}$$

we can then compute $\dfrac{\partial \phi}{\partial \rho}$

Lets define the function $p(\rho)$

$$\boxed{p(\rho) = -\int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$$

Differentiating gives

$$\frac{\partial p}{\partial \rho}(\rho) = -\rho \frac{\partial \phi}{\partial \rho}(\rho)$$

with $\rho = \rho(\vec{x})$ $\quad \frac{\partial p}{\partial \rho} = \vec{\nabla} p \cdot \frac{\partial \vec{x}}{\partial \rho}$, $\quad \frac{\partial \phi}{\partial \rho} = \vec{\nabla} \phi \cdot \frac{\partial \vec{x}}{\partial \rho}$

it becomes:

$$\boxed{\frac{\vec{\nabla} p}{\rho} = -\vec{\nabla} \phi}$$

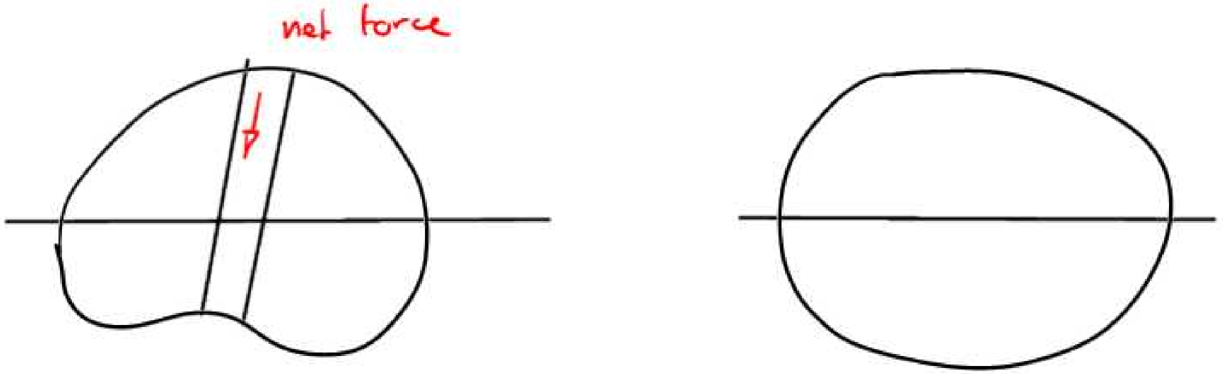
Which is the equation of equilibrium for a barotropic fluid.

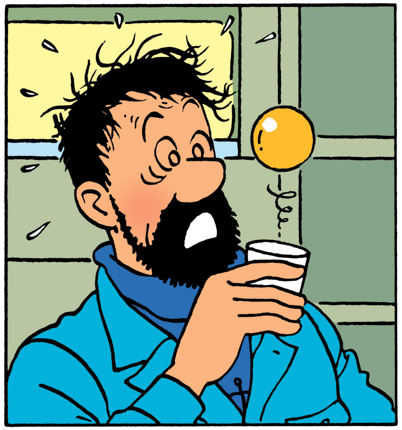
Conclusion

I. An ergodic stellar system is analog to a gasous barothrope.

II. An ergodic isolated stellar system is <u>spherical.</u>

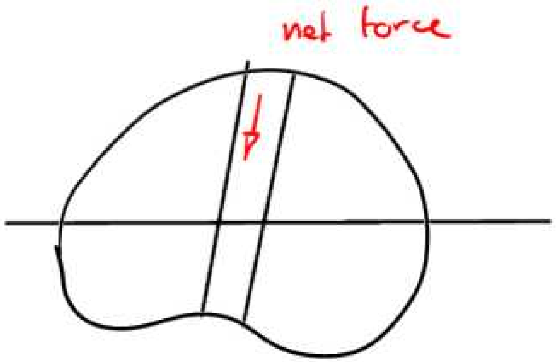
As an isolated finite, static, self-gravitating barotropic fluid must be spherical. (Lichtenstein's theorem)

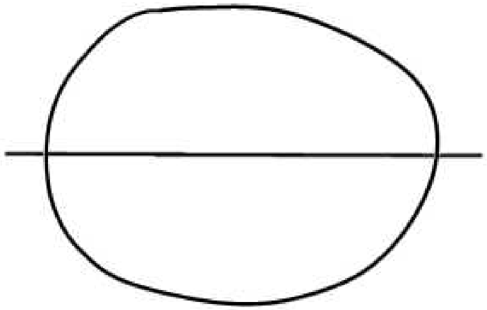


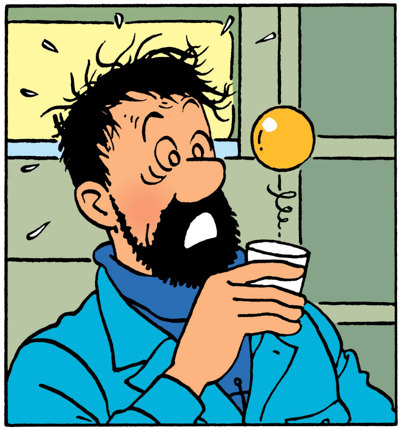


As a stellar system with an ergodic DF satisfies the same equations, it must be spherical

As an isolated finite, static, self-gravitating barotropic fluid must be spherical. (Lichtenstein's theorem)







Theorem

Any isolated, finite, stellar system with an ergodic distribution function must be spherical.

Self-consistent spherical models with ergodic DFs

Distribution function for spherical systems (Ergodic DFs)

isothropic velocity field

Goal provide a self-consistent model for a spherical stellar system

ex:
- elliptical galaxy
- galaxy cluster
- globular cluster

self-consistent = the density obtained from the DF is the one that generates the potential

i.e. is a solution of the Poisson equation

$$\boxed{\rho(\vec{x}) = N m \int d^3v \, f\Big(\underbrace{\tfrac{1}{2}v^2 + \phi(\vec{x})}_{H(\vec{x},\vec{v})}\Big) \equiv \frac{1}{4\pi G} \nabla^2 \phi(\vec{x})}$$

assumptions : only one type of stars (one stellar population)

i.e. all stars are modeled through the same DF.

Distribution function for spherical systems

- <u>Method</u> 1

  - from $\rho(r)$ $\phi(r)$ $\rightarrow$ get $f(\varepsilon) \equiv f(\frac{1}{2}v^2 + \phi(r))$

- <u>Method</u> 2

  - assume $f(\varepsilon)$ $\rightarrow$ get $\rho(r)$

Spherical systems defined by DFs

Equilibria of collisionless systems

DFs from mass distribution

Determination of the DF from the mass distribution

- We assume that $\rho(r)$ and $\phi(r)$ are known functions related together by the Poisson equation : $\nabla^2\phi = 4\pi G \rho$

- The density is related to the DF : $\nu(r) = \frac{\rho(r)}{Nm} = \frac{\rho(r)}{M}$

$$\rho(r) = M\,\nu(r) = \int f(E)\, d^3\vec{v}$$

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 + \tfrac{1}{2}\dot{z}^2 + \phi(r)$$

$$= \tfrac{1}{2}v^2 + \phi(r)$$

$$= \int_0^\infty dv\, 4\pi v^2 f\left(\tfrac{1}{2}v^2 + \phi(r)\right)$$

(isotropic in the velocity space)

We are thus looking for DFs f that satisfy :

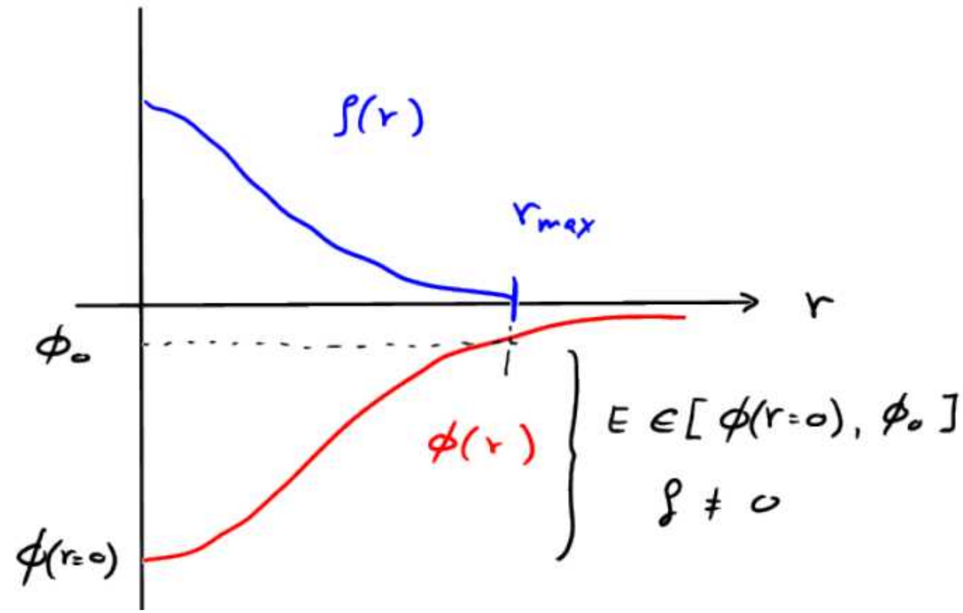
$$\boxed{\nu(r) = 4\pi \int_0^\infty dv\, v^2 f\left(\tfrac{1}{2}v^2 + \phi(r)\right)}$$

Density and potential

- $\rho(r)$ $\qquad \rho(r > r_{max}) = 0$
- $\phi(r)$ $\qquad$ no limit

Goal: find $\rho = \rho(E)$ with

$\rho = 0$ if $r > r_{max}$



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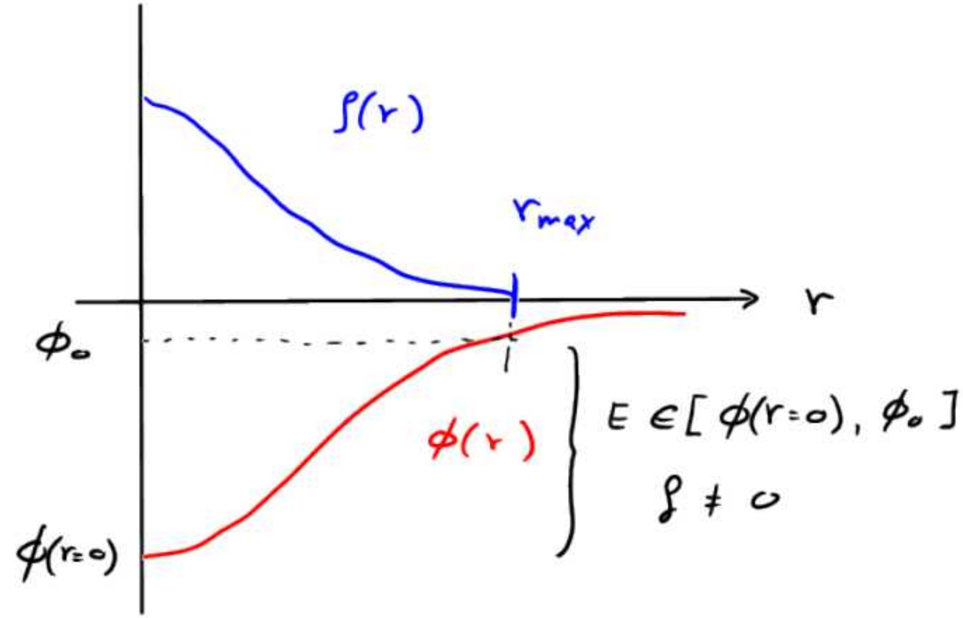


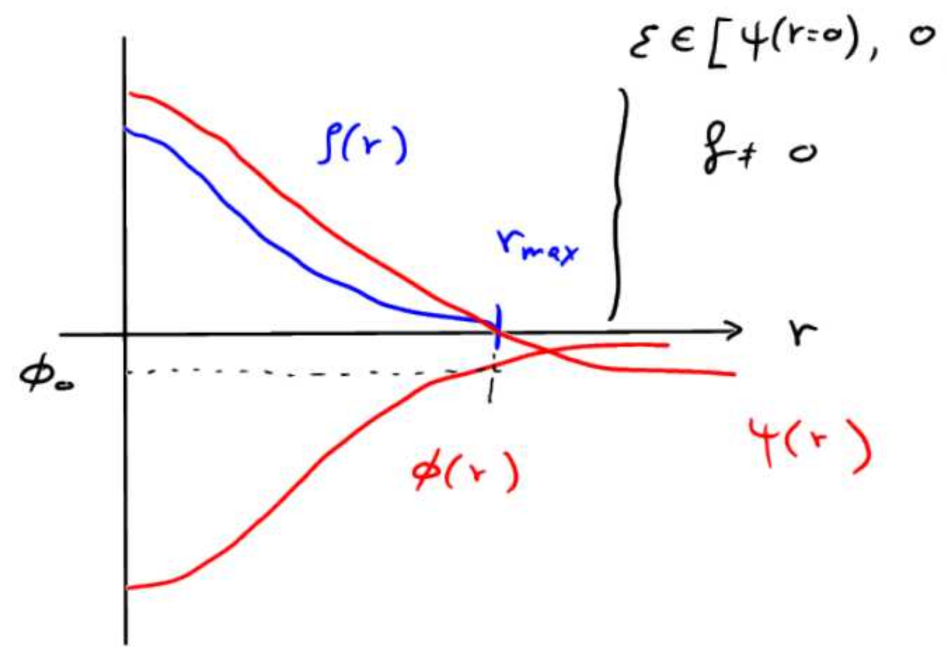
$$\boxed{\begin{array}{l}\text{Goal: find } \rho = \rho(\varepsilon) \text{ with} \\ \qquad \rho = 0 \quad \text{if} \quad r > r_{max}\end{array}}$$

Idea $\quad$ new variables

relative potential

$$\begin{cases} \psi = -(\phi - \phi_0) = -\phi + \phi_0 \\ \mathcal{E} = -(H - \phi_0) = -H + \phi_0 \end{cases}$$

relative energy $\qquad = \psi - \frac{1}{2}v^2$



$$\boxed{\rho \rightarrow \rho(\varepsilon)} \qquad \begin{cases} \mathcal{E} > 0 & \rho > 0 \\ \mathcal{E} \leq 0 & \rho = 0 \end{cases}$$

Back to the density

$$\boxed{\nu(r) = 4\pi \int_0^\infty dV\, V^2\, \underbrace{f\left(\tfrac{1}{2}V^2 + \phi(r)\right)}_{f(\mathcal{E}) \,\equiv\, f(4 - \frac{1}{2}V^2)}}$$

But $f(\mathcal{E}) = 0$ if $\mathcal{E} \leq 0$ i.e $4 - \frac{1}{2}V^2 < 0$

i.e $V > \sqrt{24}$

So, we can limit the integral to:

$[0, \sqrt{24}]$

$$\boxed{\nu(r) = 4\pi \int_0^{\sqrt{24}} dV\, V^2\, f\left(4 - \tfrac{1}{2}V^2\right)}$$

Now, lets integrate over $\mathcal{E}$, rather than v

as $\quad \mathcal{E} = \psi - \frac{1}{2} v^2$

$$v = \sqrt{2(\psi - \mathcal{E})} \quad \text{and} \quad dv = \frac{-1}{\sqrt{2(\psi - \mathcal{E})}} \, d\mathcal{E}$$

$$\nu(r) = 4\pi \int_0^{\sqrt{2\psi}} dv \, v^2 f\left(\psi - \tfrac{1}{2} v^2\right)$$

becomes

$$\nu(r) = 4\pi \int_{\psi \, \binom{v=0}{\mathcal{E}=\psi}}^{0 \, \binom{v=\sqrt{2\psi}}{\mathcal{E}=0}} d\mathcal{E} \; 2(\psi - \mathcal{E}) \, f(\mathcal{E}) \, \frac{-1}{\sqrt{2(\psi - \mathcal{E})}}$$

$$= 4\pi \int_0^{\psi} d\mathcal{E} \, \sqrt{2(\psi - \mathcal{E})} \, f(\mathcal{E})$$

- if ψ is a monotonic function of r (typical potential)

$$\psi(r) \to r(\psi) \Rightarrow \nu(r) = \nu(r(\psi)) = \nu(\psi)$$

and thus

$$\frac{1}{\sqrt{8}\,\pi} \nu(\psi) = \int_0^{\psi} d\mathcal{E} \sqrt{\psi - \mathcal{E}}\, f(\mathcal{E})$$

Derivating with respect to ψ (not trivial), we get

$$\boxed{\frac{1}{\sqrt{8}\,\pi} \frac{\partial \nu(\psi)}{\partial \psi} = \int_0^{\psi} d\mathcal{E} \frac{f(\mathcal{E})}{\sqrt{\psi - \mathcal{E}}}}$$

Abel integral

Solution : Eddington formula

$$f(\varepsilon) = \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\varepsilon} \left[\int_0^{\varepsilon} \frac{d\Psi}{\sqrt{\varepsilon - \Psi}} \frac{d\nu}{d\Psi} \right]$$

or

$$f(\varepsilon) = \frac{1}{\sqrt{8}\,\pi^2} \left[\int_0^{\varepsilon} \frac{d\Psi}{\sqrt{\varepsilon - \Psi}} \frac{d^2\nu}{d\Psi^2} + \frac{1}{\sqrt{\varepsilon}} \left(\frac{d\nu}{d\Psi} \right)_{\Psi=0} \right]$$

Note : $f(\varepsilon) > 0$ only if $\displaystyle\int_0^{\varepsilon} \frac{d\Psi}{\sqrt{\varepsilon - \Psi}} \frac{d\nu}{d\Psi}$

is an increasing function of ε !

How using this formula ?

$$\boxed{f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\mathcal{E}}\left[\int_0^{\mathcal{E}} \frac{d\psi}{\sqrt{\mathcal{E}-\psi}} \frac{d\nu}{d\psi} \right]}$$

- We start from a given $\rho(r)$, $\phi(r)$

① get r_{max} and compute $\phi_0 = \phi(r_{max})$

② a) get $\nu(r) = \rho(r)/M$

$\psi(r) = -\phi(r) + \phi_0$

b) and $\nu = \nu(\psi)$ if $\psi(r)$ may be inverted

③ if $\frac{\partial \nu}{\partial \psi}$ is analytical , compute $f(\mathcal{E})$ (Eddington's formula)

④ $f(x, v) = f(\mathcal{E}) = f(\phi_0 - \mathcal{E}) = f(\frac{1}{2}v^2 + \phi)$

Note ②a and ③ may be performed numerically

Example : Hernquist model

- $$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)}$$

$$M(r) = 2\pi \rho_0 a^3 \frac{(r/a)^2}{(1+r/a)^2}$$

$$M = M(\infty) = 2\pi \rho_0 a^3$$

- $$\phi(r) = -2\pi G \rho_0 \frac{a^2}{(1+r/a)}$$

The density is non-zero at $r = \infty$ $\Rightarrow$ $\phi_0 = 0$

$$\psi(r) = -\phi(r)$$

- inverting $\phi(r)$, we have

$$r/a = \frac{2\pi G \rho_0 a^2}{\psi} - 1 \underset{M = 2\pi \rho_0 a^3}{=} \frac{GM}{\psi a} - 1 \underset{\tilde{\psi} := \frac{\psi}{GM} a}{=} \frac{1}{\tilde{\psi}} - 1$$

we can now express ν as $\nu(\psi)$, eliminating r/a

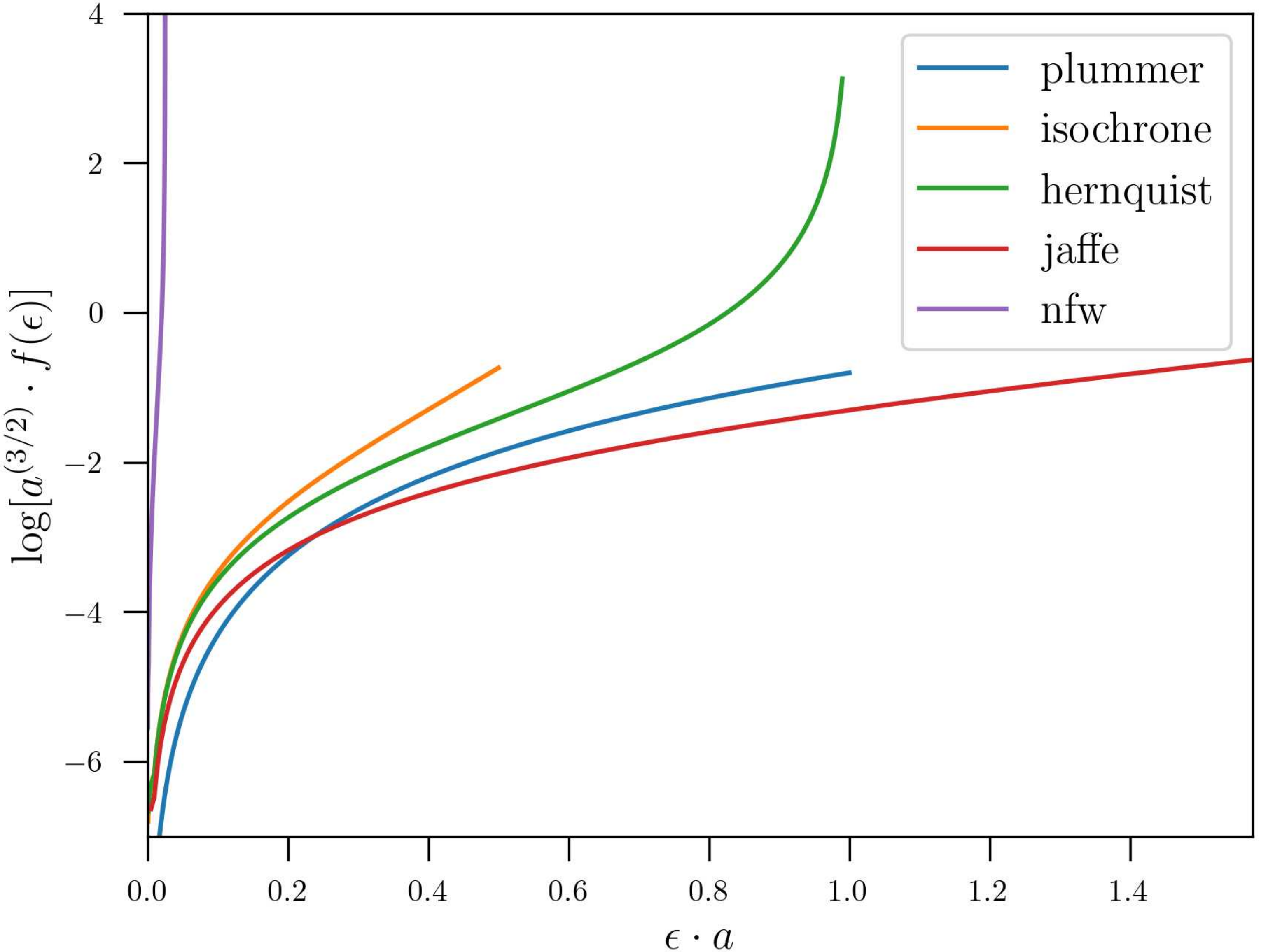
$$\nu(\psi) = \frac{\rho}{M} = \frac{1}{2\pi a^3} \frac{\tilde{\psi}^4}{1-\tilde{\psi}^4}$$

then
$$\frac{\partial \nu(\psi)}{\partial \psi} = \frac{1}{2\pi a^2 GM} \frac{\tilde{\psi}^3(4-3\tilde{\psi})}{(1-\tilde{\psi})^2}$$

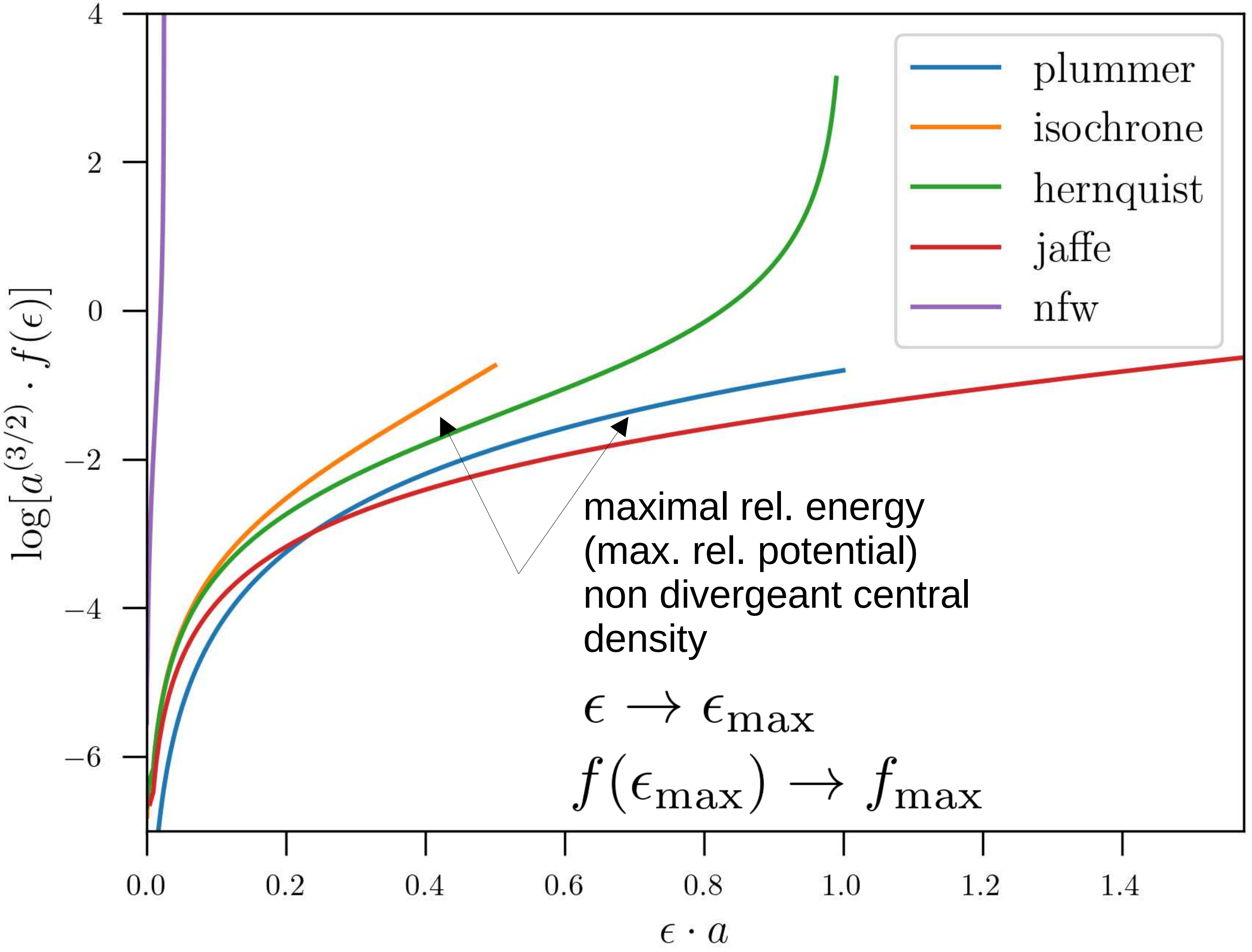
And the DF becomes, using $\tilde{\varepsilon} = -\frac{\varepsilon a}{GM}$

$$\boxed{\begin{aligned} f(\varepsilon) &= \frac{\sqrt{2}}{(2\pi)^3 (GM)^2 a} \int_0^{\varepsilon} \frac{d\psi}{\sqrt{\varepsilon-\psi}} \frac{2\tilde{\psi}^2(6-8\tilde{\psi}+3\tilde{\psi}^2)}{(1-\tilde{\psi})^3} \\ &= \frac{1}{\sqrt{2}(2\pi)^3 (GMa)^{3/2}} \frac{\sqrt{\tilde{\varepsilon}}}{(1-\tilde{\varepsilon})^2} \left[(1-2\tilde{\varepsilon})(8\tilde{\varepsilon}^2-8\tilde{\varepsilon}-3) + \frac{3\arcsin(\sqrt{\tilde{\varepsilon}})}{\sqrt{\tilde{\varepsilon}(1-\tilde{\varepsilon})}} \right] \end{aligned}}$$

<u>Note</u>: Proceeding similary, it is possible to compute the DF for others spherical potentials



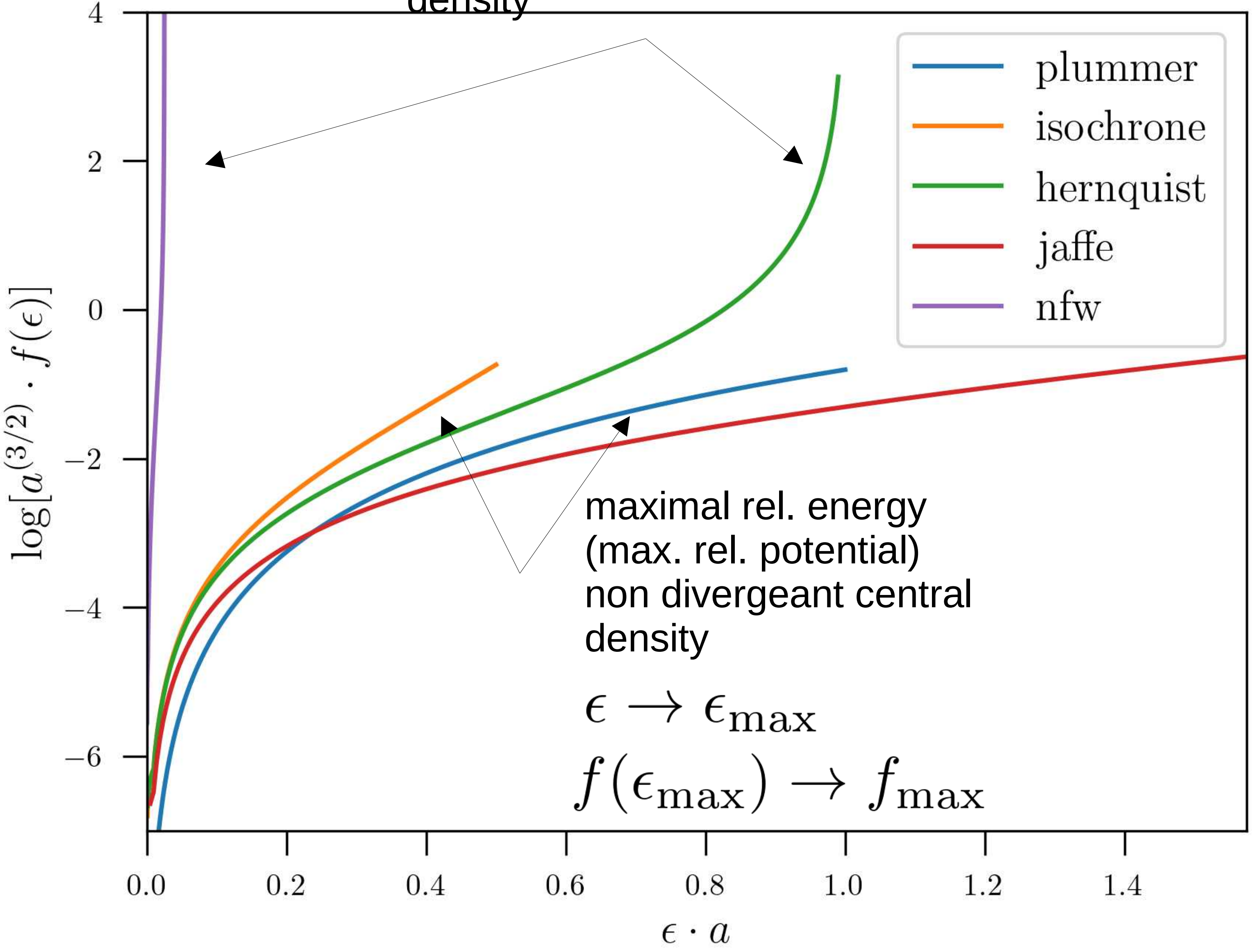
Note: Proceeding similary, it is possible to compute the DF for others spherical potentials



Note: Proceeding similary, it is possible to compute the DF for others spherical potentials

maximal rel. energy
(max. rel. potential)
divergeant central
density

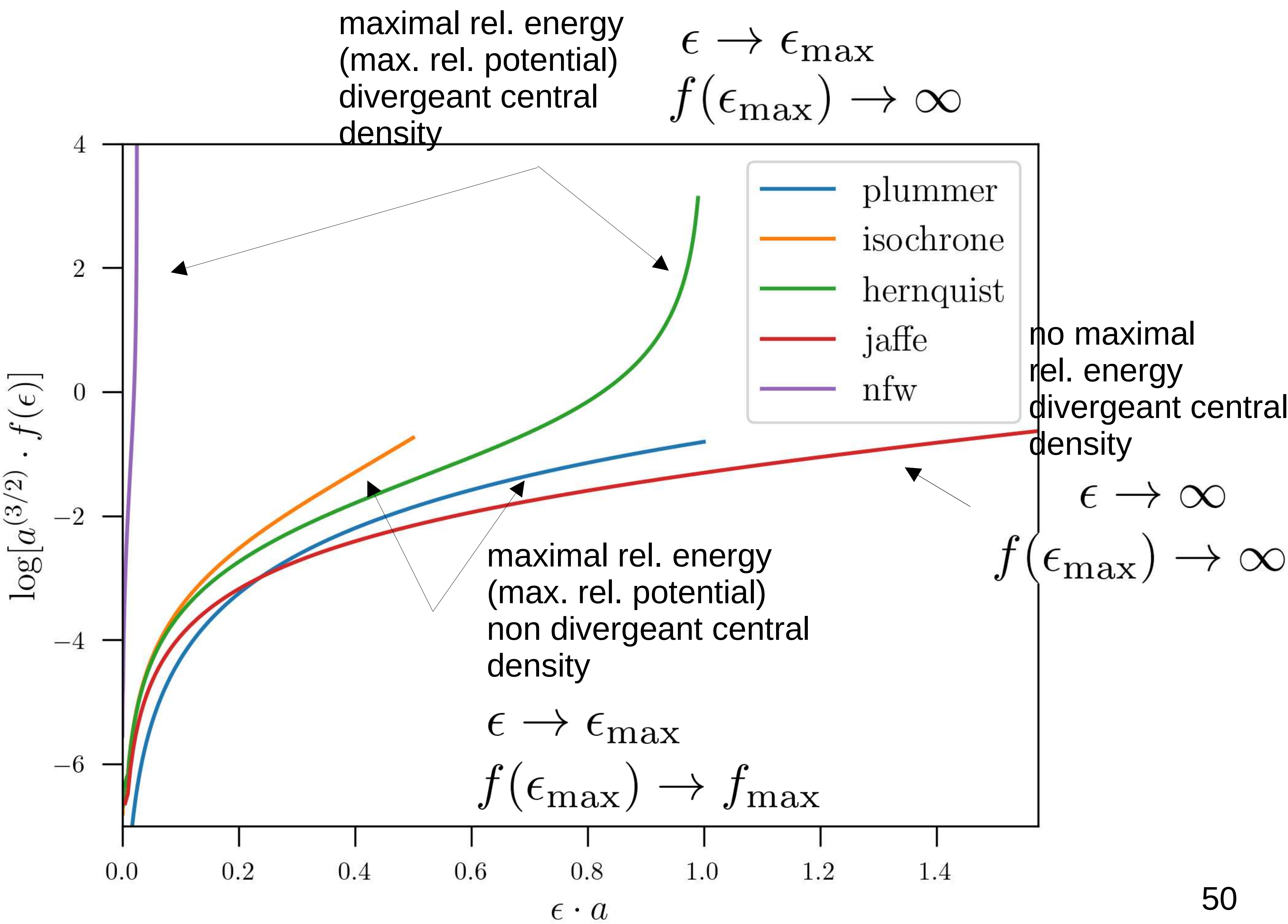
$$\epsilon \rightarrow \epsilon_{\max}$$
$$f(\epsilon_{\max}) \rightarrow \infty$$



maximal rel. energy
(max. rel. potential)
non divergeant central
density

$$\epsilon \rightarrow \epsilon_{\max}$$
$$f(\epsilon_{\max}) \rightarrow f_{\max}$$

Note: Proceeding similary, it is possible to compute the DF for others spherical potentials



Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}$$

$$\rho(r) = \frac{3M}{4\pi b^3}\left(1 + \frac{r^2}{b^2}\right)^{-5/2}$$

Isochrone model

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M\,\frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(b + 3\sqrt{b^2 + r^2})}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 + r^2)^{3/2}}$$

Jaffe model

$$\Phi(r) = -4\pi G\rho_0 a^2\,\ln(1 + a/r)$$

$$\rho(r) = \frac{\rho_0}{(r/a)^2(1 + r/a)^2}$$

Hernquist model

$$\Phi(r) = -4\pi G\rho_0 a^2\,\frac{1}{2(1 + r/a)}$$

$$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)^3}$$

Equilibria of collisionless systems

Models defined from DFs

Distribution function for spherical systems

- Method 1
  - from $\rho(r)$ $\phi(r)$ $\longrightarrow$ get $f(\varepsilon) = f\left(\frac{1}{2}v^2 + \phi(r)\right)$
- Method 2
  - assume $f(\varepsilon)$ $\longrightarrow$ get $\rho(r)$

Spherical systems defined by DFs

Equilibria of collisionless systems

Models defined from DFs: Polytropes

Polythropes and Plummer models

$$\boxed{f(\mathcal{E}) = \begin{cases} F\,\mathcal{E}^{\,n-3/2} & (\mathcal{E} > 0) \\ 0 & (\mathcal{E} \leq 0) \end{cases}}$$

F, a constant

$f = 0$ if $\mathcal{E} > 0$

$\rho = 0$

Corresponding density

$$\nu(r) = 4\pi \int_0^{\sqrt{2\psi}} dv\, v^2 f\left(\psi - \tfrac{1}{2} v^2\right)$$

$\times N \cdot m$

$\nu(r) \to \rho(r)$

$$\rho(r) = 4\pi F \int_0^{\sqrt{2\psi}} dv\, v^2 \left(\psi(r) - \tfrac{1}{2} v^2\right)^{n-3/2}$$

Which leads to :

$$\boxed{\rho(r) = C_n \, \Psi(r)^n} \qquad (\text{for } \Psi > 0)$$

relation between ρ and ϕ

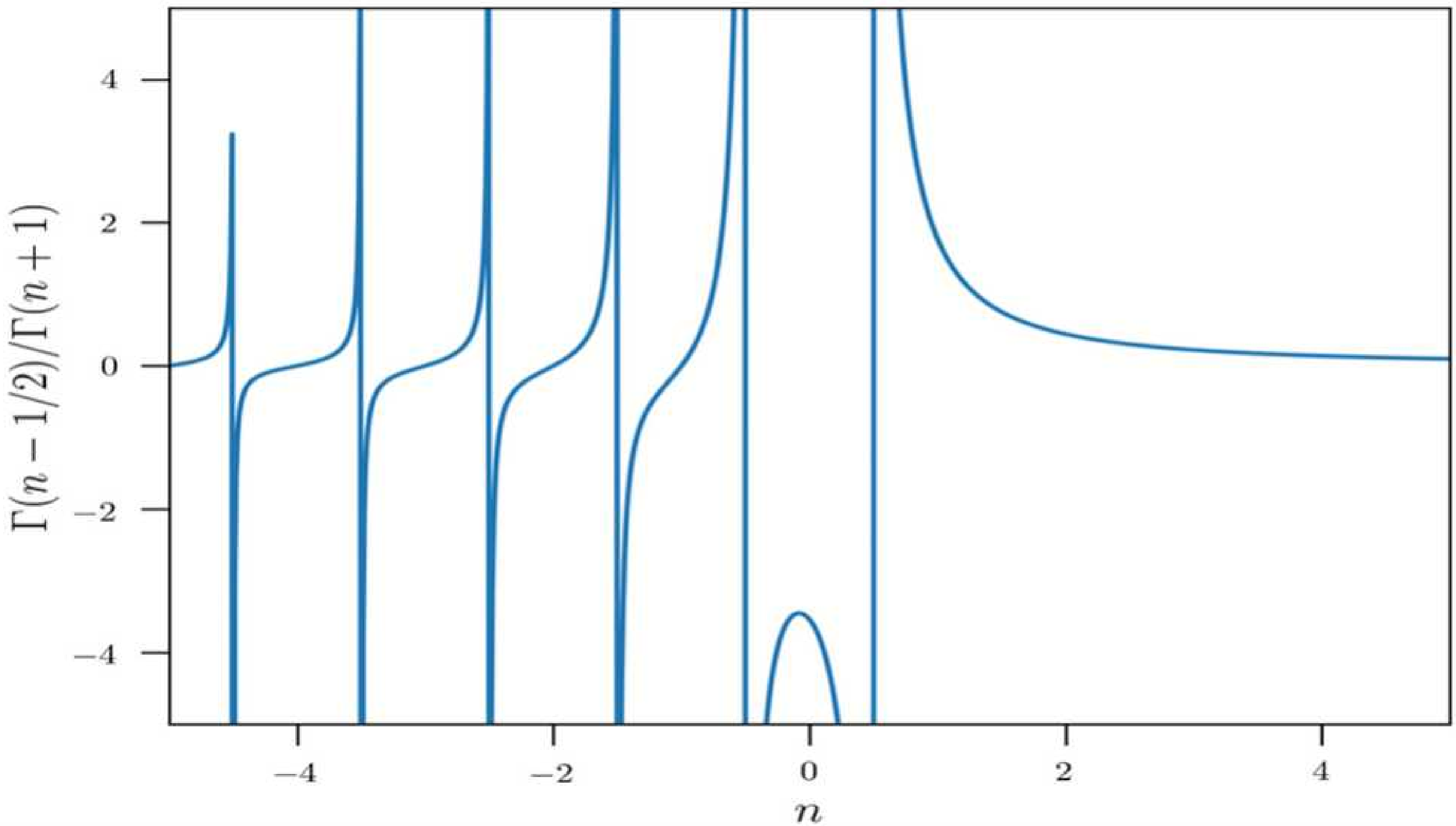
$$C_n = \frac{(2\pi)^{3/2} \left(n - \frac{3}{2}\right)! \, F}{n!} = \frac{(2\pi)^{3/2} \, \Gamma\left(n - \frac{1}{2}\right) F}{\Gamma(n+1)}$$

$$n! = \Gamma(n+1) = \int_0^\infty dt\, t^n e^{-t}$$

$$c_n \sim \frac{(n-\frac{3}{2})!}{n!} = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+1)}$$

$n = \frac{1}{2}$

$n > \frac{1}{2}$, $c_n > 0$, $f > 0$



Demonstration

$$\rho(r) = 4\pi F \int_0^{\sqrt{2\psi}} dV \, V^2 \left(\psi(r) - \tfrac{1}{2} V^2\right)^{n-3/2}$$

smart substitution : introduce the variable $\theta(V)$ such that

$$V^2 = 2\psi \cos^2\theta \quad , \quad \theta = \arccos\left(\frac{V}{\sqrt{2\psi}}\right)$$

$$2V\,dV = -4\psi \cos\theta \sin\theta \, d\theta$$

$$\Rightarrow \quad dV = -\frac{2\psi \cos\theta \sin\theta \, d\theta}{\sqrt{2\psi}\cos\theta} = -\sqrt{2\psi} \sin\theta \, d\theta$$

$$\psi - \tfrac{1}{2}V^2 = \psi - \psi\cos^2\theta = \psi \sin^2\theta$$

$$\left\{ \begin{array}{l} V = 0 \rightarrow \theta = \pi/2 \\ V = \sqrt{2\psi} \rightarrow \theta = 0 \end{array} \right.$$

$$\rho(r) = 4\pi F \int_0^{\pi/2} \left(\sqrt{2\psi} \sin\theta \, d\theta\right) \cdot \left(2\psi \cos^2\theta\right) \cdot \left(\psi \sin^2\theta\right)^{n-3/2}$$

$$= 4\pi F \int_0^{\pi/2} 2 \cdot 2^{\frac{1}{2}} \psi^{\frac{1}{2}} \psi \, \psi^{n-\frac{3}{2}} \cdot \cos^2\theta \, \sin\theta^{2n-2} \, d\theta$$

$$= 8\pi F \sqrt{2}\, \Psi^n \int_0^{\frac{\pi}{2}} \underbrace{\cos^2\theta}_{1-\sin^2} \sin\theta^{2n-2}\, d\theta$$

So, we get

$$\rho(r) = C_n \Psi(r)^n \qquad (\text{for } \Psi > 0)$$

relation between ρ and ϕ

$$C_n = \frac{(2\pi)^{3/2} \left(n - \frac{3}{2}\right)!\, F}{n!} = \frac{(2\pi)^{3/2}\, \Gamma\left(n - \frac{1}{2}\right) F}{\Gamma(n+1)}$$

COMPLEMENT

Corresponding "Pressure"

$$\boxed{P(\rho) = - \int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$$

$$\rho = C_n \, \psi^n$$

$$\psi = \frac{1}{C_n^{1/n}} \, \rho^{\frac{1}{n}} \qquad \frac{\partial \psi}{\partial \rho} = \frac{1}{C_n^{1/n}} \, \frac{1}{n} \, \rho^{\frac{1}{n}-1}$$

$$\frac{\partial \phi}{\partial \rho} = - \frac{1}{C_n^{1/n}} \, \frac{1}{n} \, \rho^{\frac{1}{n}-1}$$

$$P(\rho) = \frac{1}{C_n^{1/n}} \, \frac{1}{n} \int_0^{\rho} d\rho' \, \rho'^{\frac{1}{n}} = \frac{1}{C_n^{1/n}} \, \frac{1}{n+1} \, \rho^{\frac{1}{n}+1}$$

$$\boxed{P(\rho) = K \, \rho^{\gamma}}$$

$\equiv$ Polytropic EoS

$$\begin{cases} \gamma = \frac{1}{n} + 1 & n = \frac{1}{\gamma - 1} \\ K = \frac{1}{C_n^{1/n}} \, \frac{1}{n+1} & C_n = \left(\frac{\gamma - 1}{K \gamma} \right)^{\frac{1}{\gamma - 1}} \end{cases}$$

Conclusion

The density of a stellar system described by and ergodic DF

$$f(\epsilon) \sim \epsilon^{n-3/2}$$

Is the same as a polytropic gas sphere in hydrostatic equilibrium, with:

$$P(\rho) \sim \rho^{\gamma}$$

This is why these DFs are called polytropes.

<u>Note</u> : from $\rho(r) = C_n \Psi(r)^n$

if $\rho = \text{cte}$ $\Rightarrow$ $n = 0$

But from $C_n = \dfrac{(2\pi)^{3/2}\, \Gamma(n-\frac{1}{2})\, F}{\Gamma(n+1)}$ $\Rightarrow$ $C_n < 0$ $\rho < 0$!

① No finite ergodic stellar system is homogeneous.

② No self-gravitating homogeneous system equivalent to a self-gravitating sphere of incompressible fluid exists.

Indeed : the hydrostatic solution of an incompressible fluid of constant density requires

$$\frac{dP}{dr} = -\rho_0 \frac{d\phi}{dr} = -\frac{4}{3}\pi G \rho_0^2 r$$

not a polytropic EOS $\leftarrow$

$$P = P_0 - \frac{2}{3}\pi G \rho_0^2 r^2$$

Self-gravity !

$$\vec{\nabla}^2(\Phi) = 4\pi G\rho$$

The Poisson equation for spherical systems (with ψ)

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = -4\pi G\rho(r)}$$

$$\rho = C_n\psi^n$$

$$\rho^{\frac{n-1}{n}} = C_n^{\frac{n-1}{n}}\psi^{n-1}$$

With $\rho = C_n\psi^n$

$$\frac{d\rho}{dr} = C_n\, n\, \psi^{n-1}\frac{d\psi}{dr} = C_n\, n\left(\frac{1}{C_n}\rho\right)^{\frac{n-1}{n}}\frac{d\psi}{dr}$$

thus

$$\frac{d\psi}{dr} = \frac{1}{C_n^{1/n}\, n}\rho^{\frac{1-n}{n}}\frac{d\rho}{dr}$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{n\,C_n^{1/n}}\rho^{\frac{1-n}{n}}\frac{d\rho}{dr}\right) + 4\pi G\rho = 0}$$

or eliminating ρ, using $\rho(r) = C_n\psi(r)^n$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + 4\pi G C_n\psi^n = 0}$$

Solutions

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G c_n \Psi^n = 0}$$

A. Power laws

$$\left\{\begin{array}{l} \rho(r) \sim r^{-\alpha} \\ \Psi(r) \sim r^{-\alpha/n} \end{array}\right. \qquad \rho \sim \Psi^n$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) \sim r^{-\frac{\alpha}{n}-2}$$

Poisson

$$\underbrace{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)}_{r^{-\frac{\alpha}{n}-2}} + \underbrace{4\pi G \rho(r)}_{r^{-\alpha}} = 0$$

$$-\frac{\alpha}{n}-2 \sim -\alpha$$

$$\Rightarrow \boxed{\alpha = \frac{2n}{n-1}}$$

As the potential may not decrease faster than the Kepler potential $\frac{1}{r}$

$$\left(\Psi \sim r^{-\frac{\alpha}{n}}\right) \qquad \frac{\alpha}{n} \leq 1 \quad \Rightarrow \quad \boxed{n \geq 3}$$

B Models with finite potential and density

Define new variables $s = \frac{r}{b}$ $\qquad \psi' = \frac{\psi}{\psi_0}$

where
$$\begin{cases} b = \left(\frac{4}{3}\pi G \psi_0^{n-1} c_n\right)^{1/2} \\ \psi_0 = \psi(0) \end{cases}$$

The Poisson equation becomes

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + 4\pi G c_n \psi^n = 0}$$

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -3\psi'^n$$

Lane - Emden Equation

+ boundary conditions

$$\begin{cases} \bullet\ \psi'(0) = 1 & \text{normalisation} \\ \bullet\ \left.\frac{d\psi'}{dr}\right)_0 = 0 & \text{no force at the center (smooth)} \end{cases}$$

(In general, non trivial solutions)

Two analytical solutions $\quad n = 1, \quad n = 5$

$$\boxed{n = 1} \qquad \boxed{\frac{1}{s^2} \frac{d}{ds} \left(s^2 \frac{d\psi'}{ds} \right) = -3\psi'}$$

linear Helmholtz Equation

$$\psi'(s) = \begin{cases} \dfrac{\sin(\sqrt{3}\, s)}{\sqrt{3}\, s} & s < \dfrac{\pi}{\sqrt{3}} \\[2ex] \dfrac{\pi}{\sqrt{3}\, s} - 1 & s \geq \dfrac{\pi}{\sqrt{3}} \end{cases}$$

Two analytical solutions $\quad n=1, \quad n=5$

$\boxed{n = 1}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -3\psi'}$$

linear Helmholtz Equation

$$\psi'(s) = \begin{cases} \dfrac{\sin(\sqrt{3}\,s)}{\sqrt{3}\,s} & s < \dfrac{\pi}{\sqrt{3}} \\[2ex] \dfrac{\pi}{\sqrt{3}\,s} - 1 & s \geq \dfrac{\pi}{\sqrt{3}} \end{cases}$$

UNPHYSICAL SOLUTION

⚠ $n = 1 < 3$

non physical solution

$$\boxed{n = 5} \qquad \boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'^5}$$

consider $\quad \psi'(s) = \dfrac{1}{\sqrt{1+s^2}}$

The Poisson Equation becomes

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi'}{ds}\right) = -\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}}\right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\longrightarrow$ $\psi'(s)$ is a solution !

$$\boxed{n = 5} \qquad \boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'^5}$$

consider $\psi'(s) = \dfrac{1}{\sqrt{1+s^2}}$

The Poisson Equation becomes

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}}\right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\rightarrow$ $\psi'(s)$ is a solution !

and corresponds to the <u>Plummer model</u>

$$\boxed{\phi(r) = -\frac{GM}{\sqrt{r^2+a^2}}} \qquad \boxed{\rho(r) = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-5/2}}$$

Then : what do we learn concerning the Plummer model ?

We have access to its DF :

$$f(\mathcal{E}) \begin{cases} \sim \mathcal{E}^{n-3/2} \sim \left(\frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 \right)^{7/2} \\ = 0 \quad \text{if} \quad \frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 < 0 \end{cases}$$

We have access to the kinematics structure :

① Velocity distribution function

$$P_r(v) = \frac{f(\frac{1}{2}v^2 + \phi(r))}{\nu(r)} \sim \underbrace{\left(1+\frac{r^2}{a^2}\right)^{5/2}}_{\frac{1}{\rho}} \underbrace{\left(\frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 \right)^{7/2}}_{\mathcal{E}^{7/2}}$$

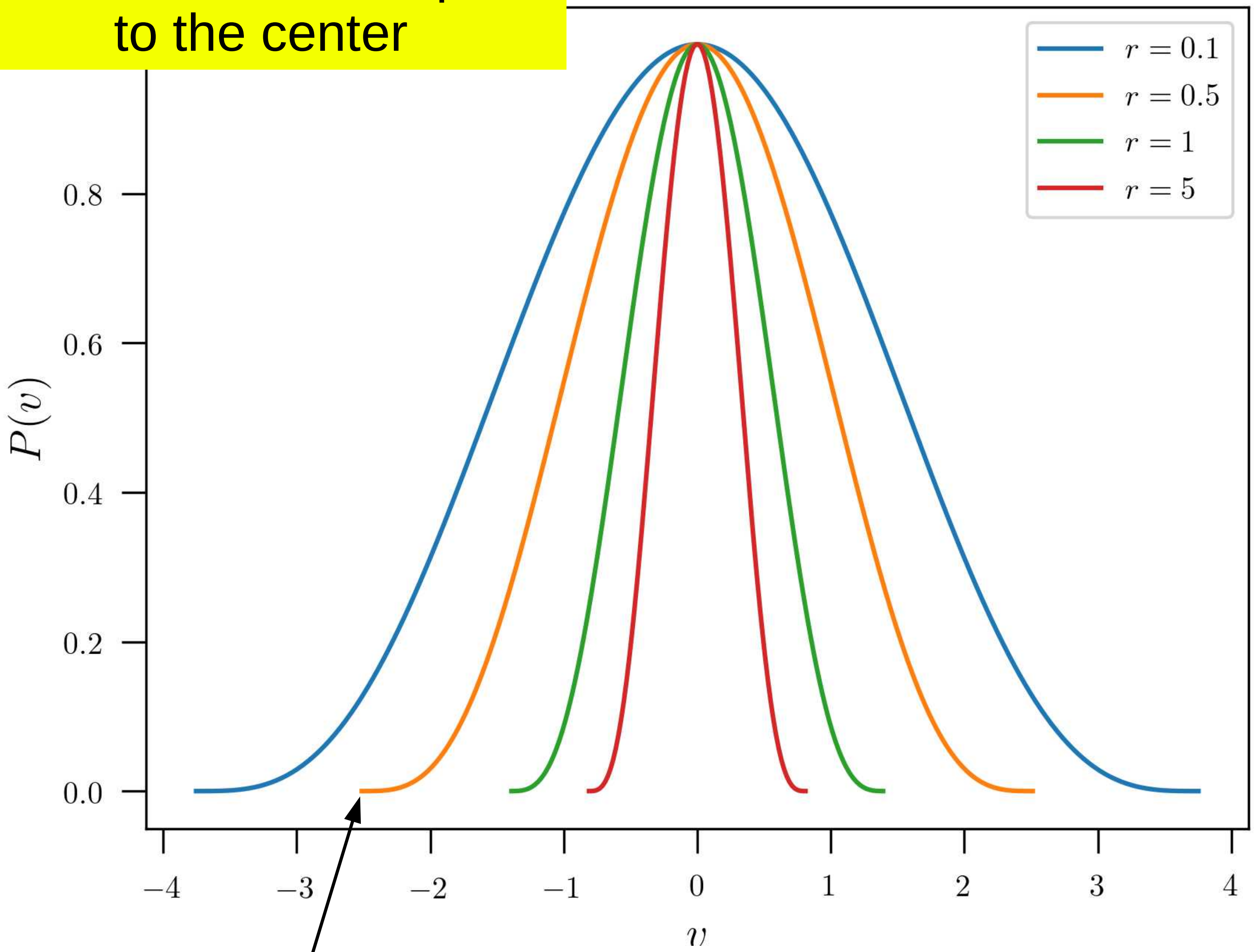
② Velocity dispersion

$$\sigma^2 = 4\pi \frac{1}{\nu(x)} \int_0^{v_{max} = \sqrt{2\psi}} v^4 f\left(\tfrac{1}{2}v^2 + \phi(\vec{x})\right) dv$$

$$= 4\pi \frac{1}{\nu(r)} \int_0^{v_{max}} v^4 \left(\frac{1}{2} v^2 - \frac{GM}{\sqrt{r^2+a^2}} \right)^{7/2} dv$$

The Plummer velocity distribution function

Normalized with respect to the center

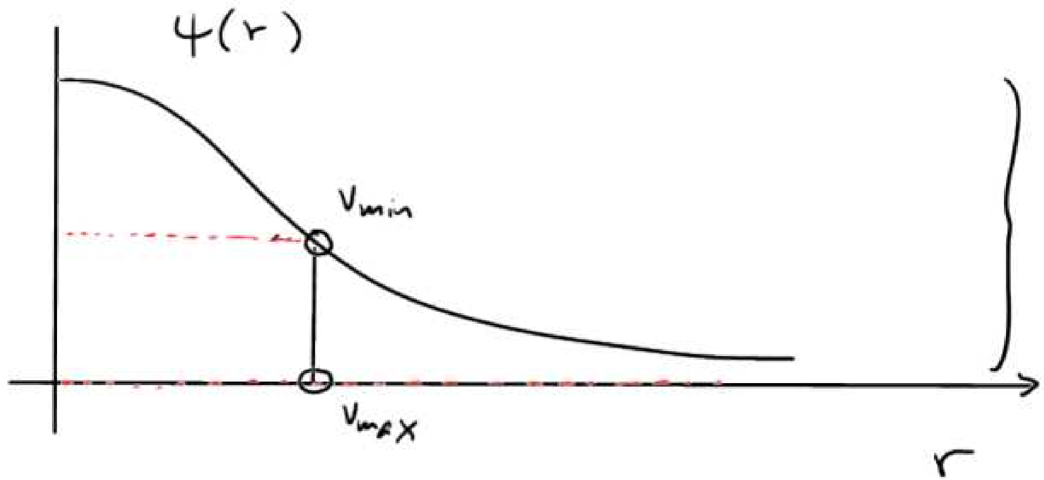


$$\forall r, \exists\, v_{\rm max} \,\text{such that}\, \epsilon > 0 \Rightarrow f = 0$$

Interpretation

$$P_r(v) = \begin{cases} \left(\dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 \right)^{7/2} & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases}$$

$$\mathcal{E} = \psi - \frac{1}{2} v^2$$



$\mathcal{E} = \psi(0)$

$\mathcal{E} = 0$

in r, the minimum velocity is $V_{min} = 0$

orbits with $r_{max} = r$, $V(r_{max}) = 0$

the maximum velocity is $V_{max} = \sqrt{2\psi(r)}$

orbits with $\mathcal{E} = 0$ $(r_{max} = \infty)$

The End