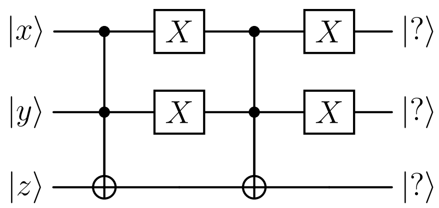
Midterm exam 2025
Quantum Computation

Please pay attention to the presentation of your answers! (2 points)

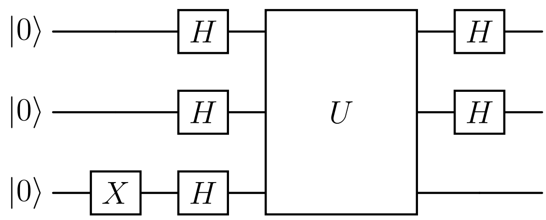
Exercise 1 *Outputs of quantum circuits (12 points)*

a) Consider the following 3-qubit quantum circuit U:



What is the output of the circuit U when the input is an element $|x, y, z\rangle$ of the computational basis?

b) Consider now the following circuit:



What is/are the possible output(s) of the first 2 qubits of the circuit (i.e., the ones following the gates H) and their corresponding probabilities?

c) How would these output probabilities be modified if the circuit U were replaced by a single Toffoli gate?

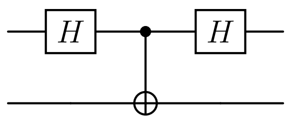
Exercise 2 *True or false? (16 points)*

For each question, 1 pt for the correct answer, 3 pts for the justification. If you think the statement is correct, then prove it; otherwise, provide a counter-example.

a) Let $|\varphi\rangle$ be a single qubit state and $|\varphi\rangle \otimes |\varphi\rangle$ be the (product) state of two qubits. A measurement in the computational basis is performed on the two qubits. Then the probability of observing the two qubits in the same state is necessarily greater than or equal to the probability of observing them in different states.

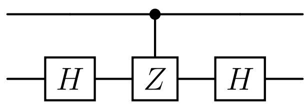
b) Classically, the output of a XOR gate is always equal to 0 if its two inputs are identical. It is also the case that the output target qubit of a CNOT gate is always $|0\rangle$ if its two input qubits are in a product state $|\varphi\rangle \otimes |\varphi\rangle$.

c) If the input to the following circuit:



is in an entangled state, then so is its output.

d) The following circuit:

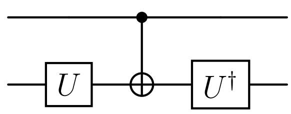


is equivalent to a C-NOT gate.

Exercise 3 *Matrix representation (5 points)*

Let $U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$ be the matrix representation of the 1-qubit gate U.

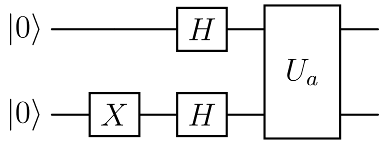
What is the matrix representation of the following 2-qubit gate? (in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$)



Exercise 4 *Communication with quantum bits (15 points)*

Consider the following problem: Alice is in possession of a 2-bit vector $a = (a_0, a_1) \in \{0,1\}^2$ and Bob is in possession of another 2-bit vector $b = (b_0, b_1) \in \{0,1\}^2$. They would like to exchange information about the vectors a and b.

To this end, Alice builds the following 2-qubit circuit:



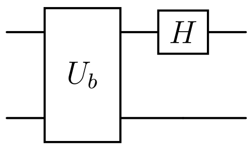
where the action of the gate U_a on a basis state $|x\rangle \otimes |y\rangle$ is described as:

$$U_a\left(|x\rangle \otimes |y\rangle\right) = |x\rangle \otimes |y \oplus a_x\rangle$$

Alice then sends the 2-qubit output state $|\psi\rangle$ of the above circuit to Bob.

a) Compute the state $|\psi\rangle$.

Bob then uses the state $|\psi\rangle$ as input to the following circuit:



where the action of the gate U_b is similar to the action of the gate U_a described above.

b) Compute the 2-qubit output state $|\psi'\rangle$ of the above circuit.

Bob then measures the first qubit (that is, the top qubit) of the output state $|\psi'\rangle$.

c1) What can he deduce on the relation between the 2-bit vectors a and b if the measured state is $|0\rangle$?

c2) And likewise, what can he deduce on the relation between a and b if the measured state is $|1\rangle$?