Equilibria of collisionless systems

1st part

Outlines

Weak bars

- the Lindblad resonances
- orbit families in realistic bars

The collisonless Boltzmann equation

- The distribution function (DF) of stellar systems
- The Collisionless Boltzmann equation
- Limitations

Relations between DFs and observables

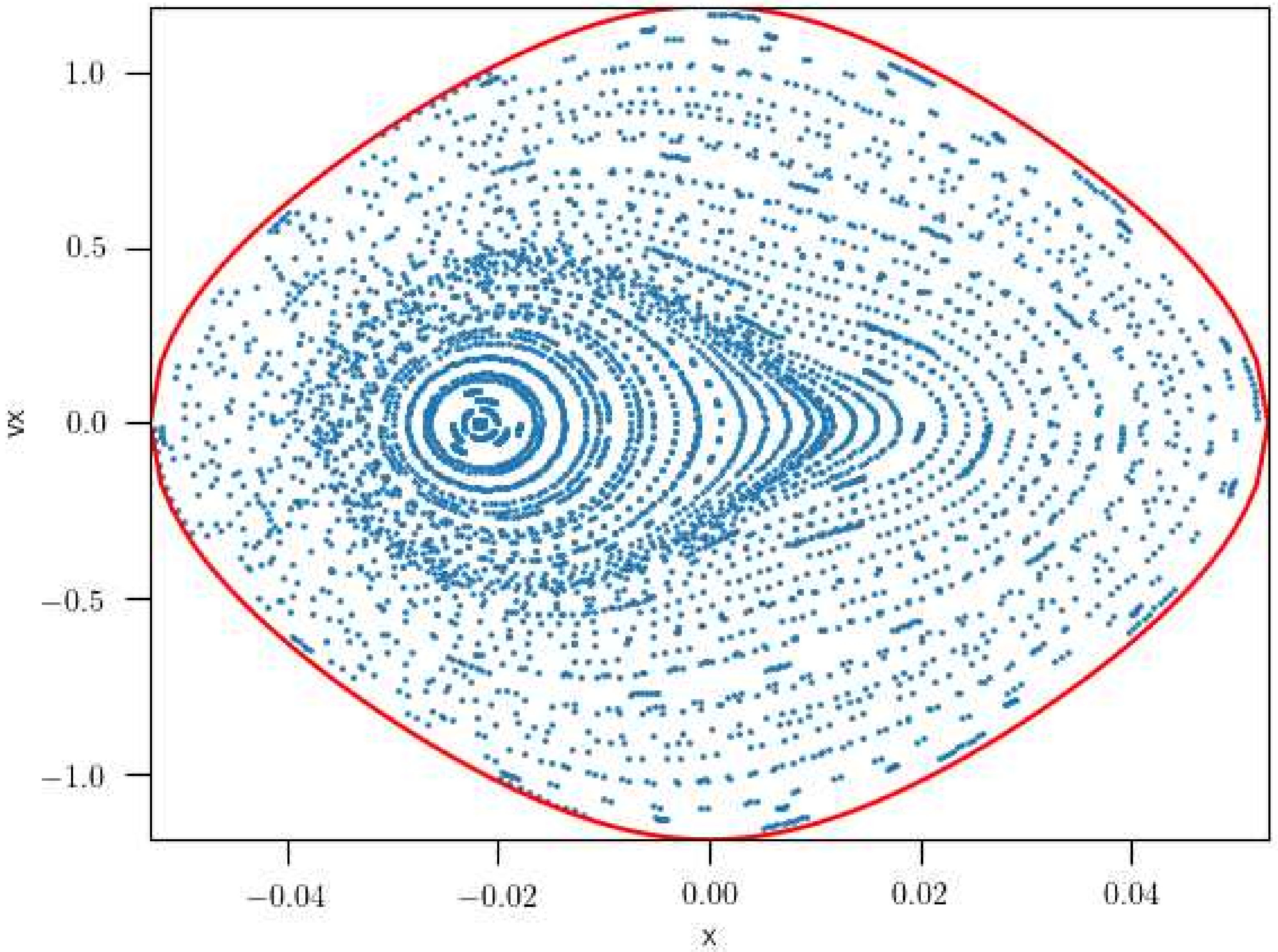
- Density, velocity distribution function, mean velocity, velocity dispersion

The Jeans theorems

- Steady-state solutions of the Collisionless Boltzmann equation

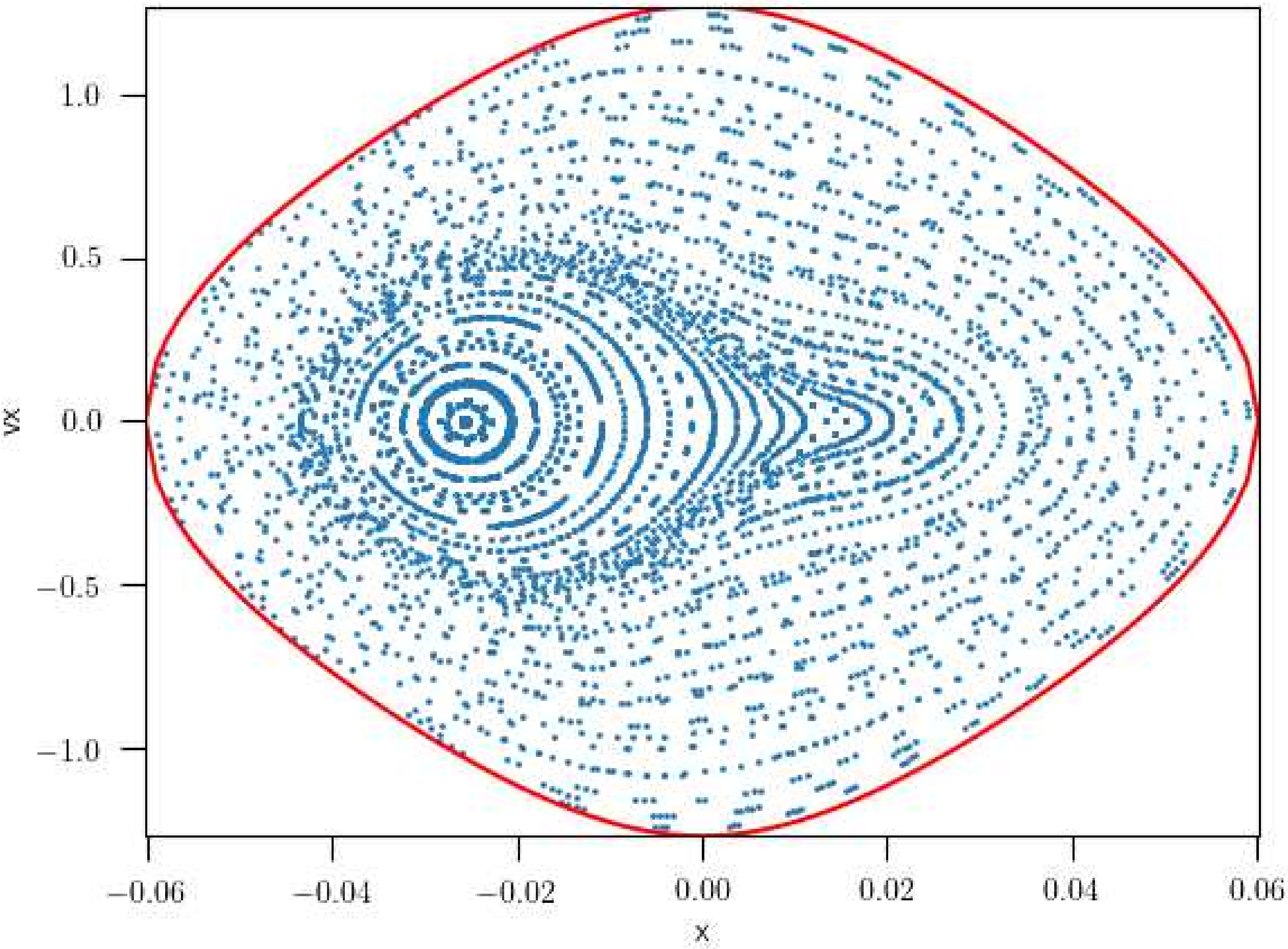
Orbits
in weak rotating bars

$$E = -2.8$$



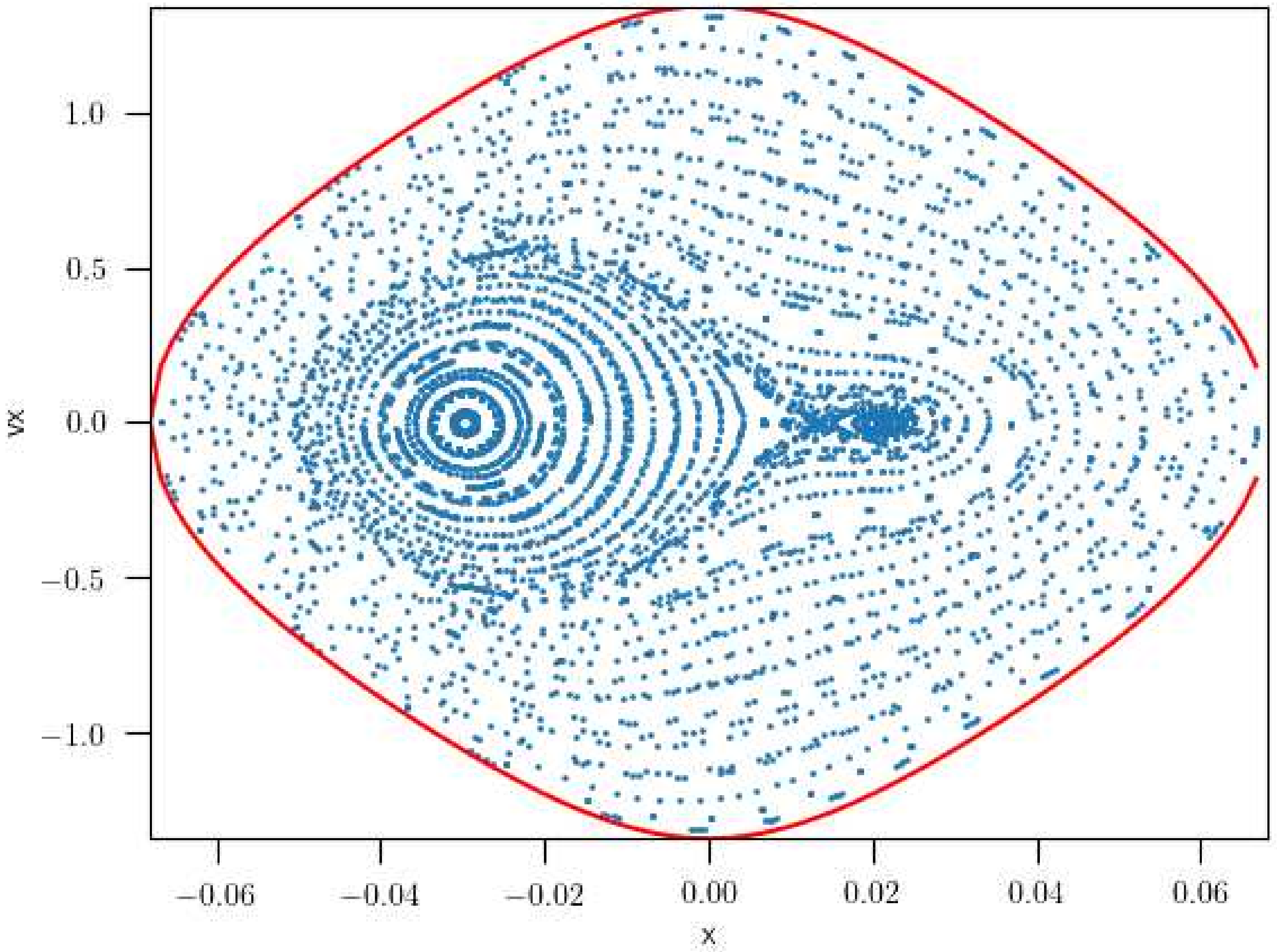

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.8 --norbits 50
```

$$E = -2.7$$



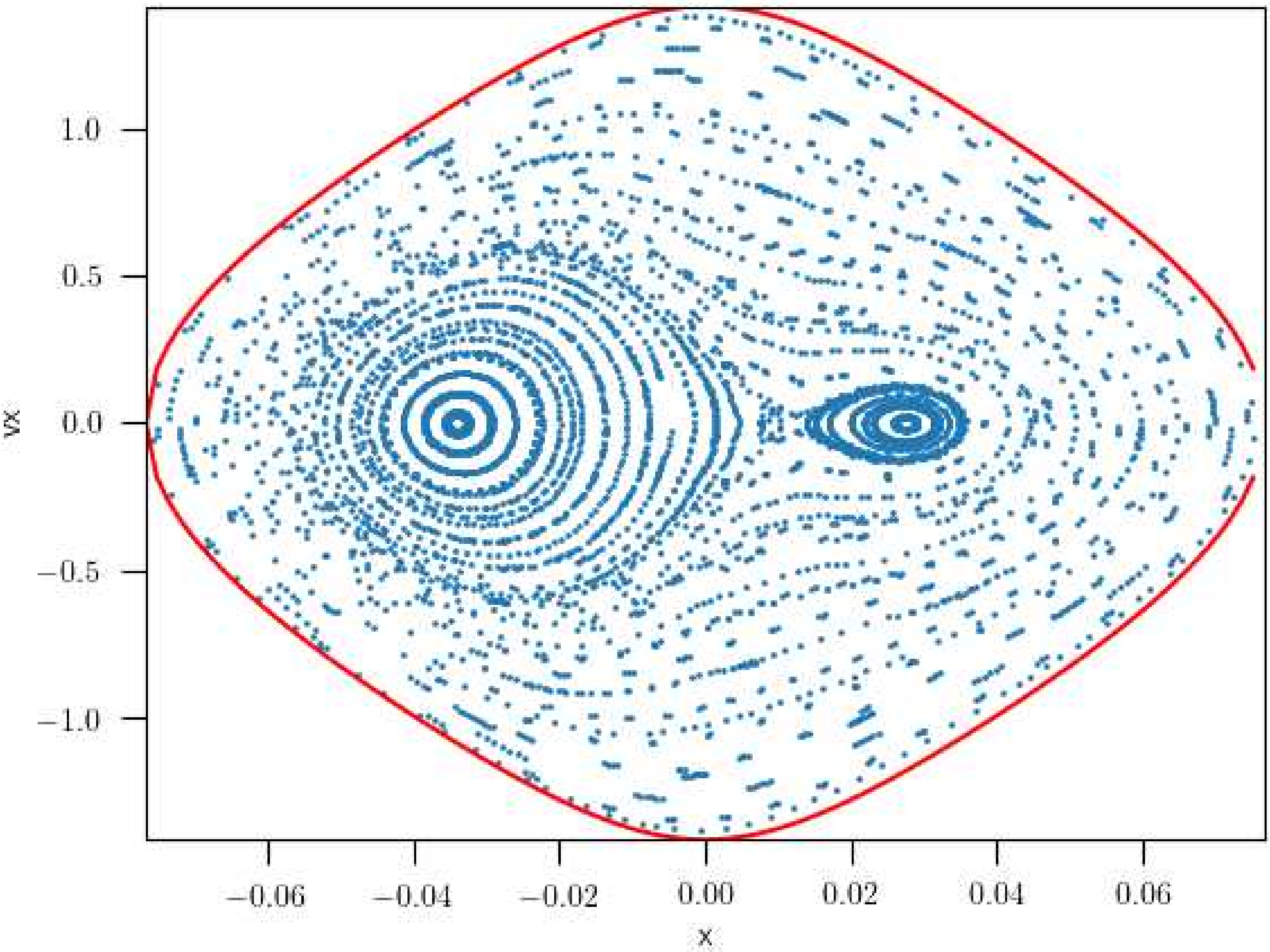

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.7 --norbits 50
```

$E = -2.6$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.6 --norbits 50
```

$$E = -2.5$$




```
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```
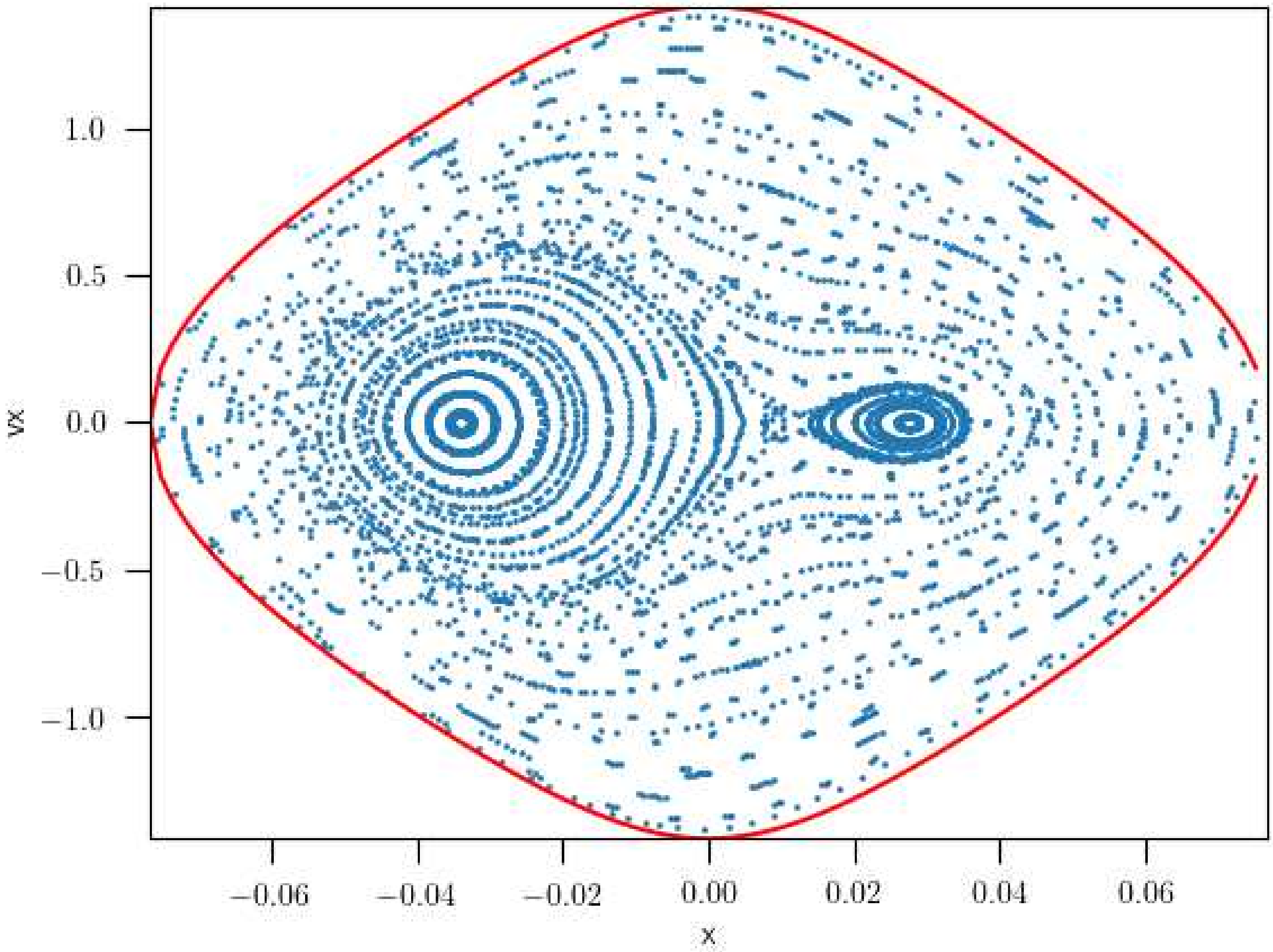
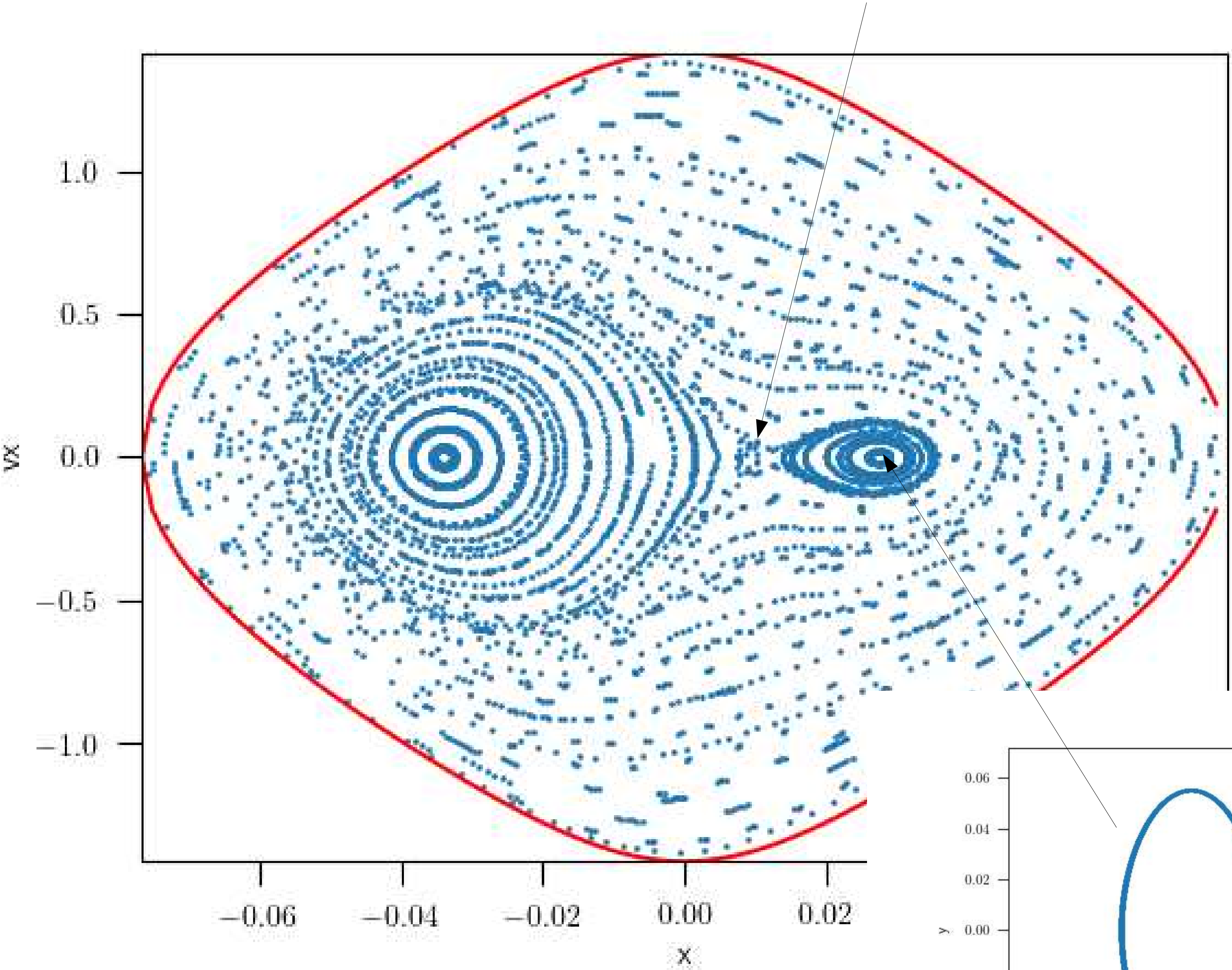
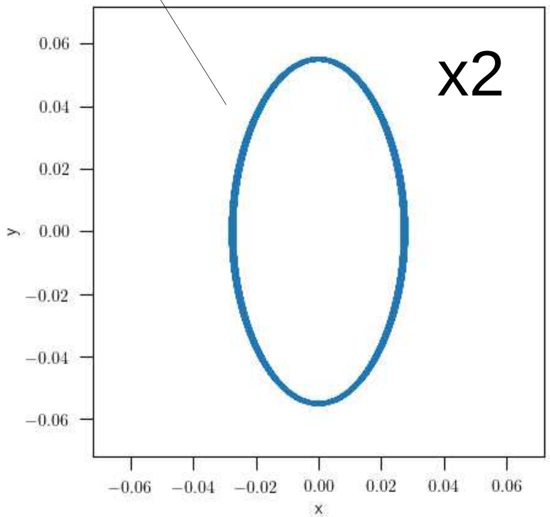

$$E = -2.5$$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

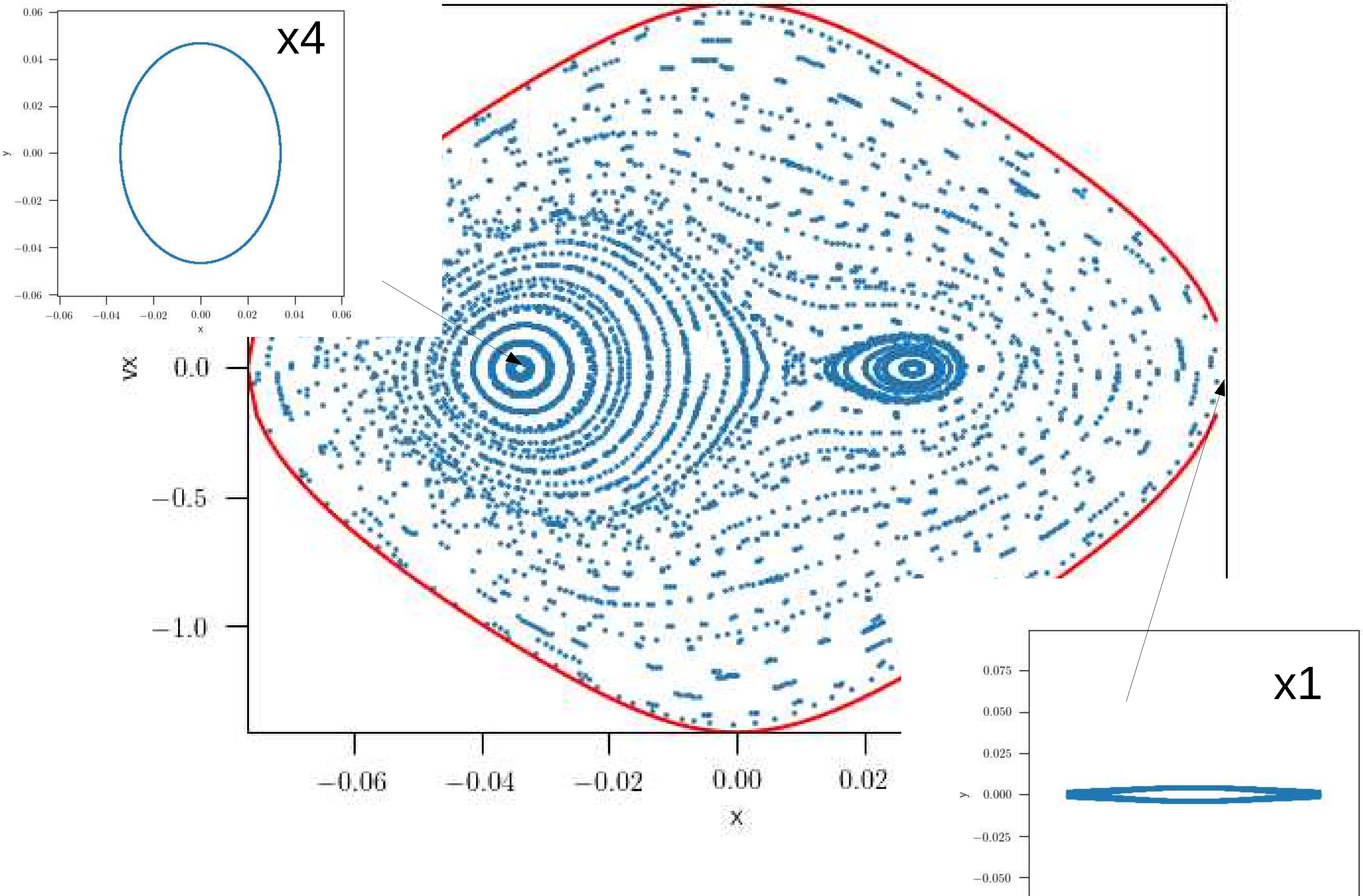
Bifurcation : apparition of x2 (stabe)/x3 (unstable) orbits





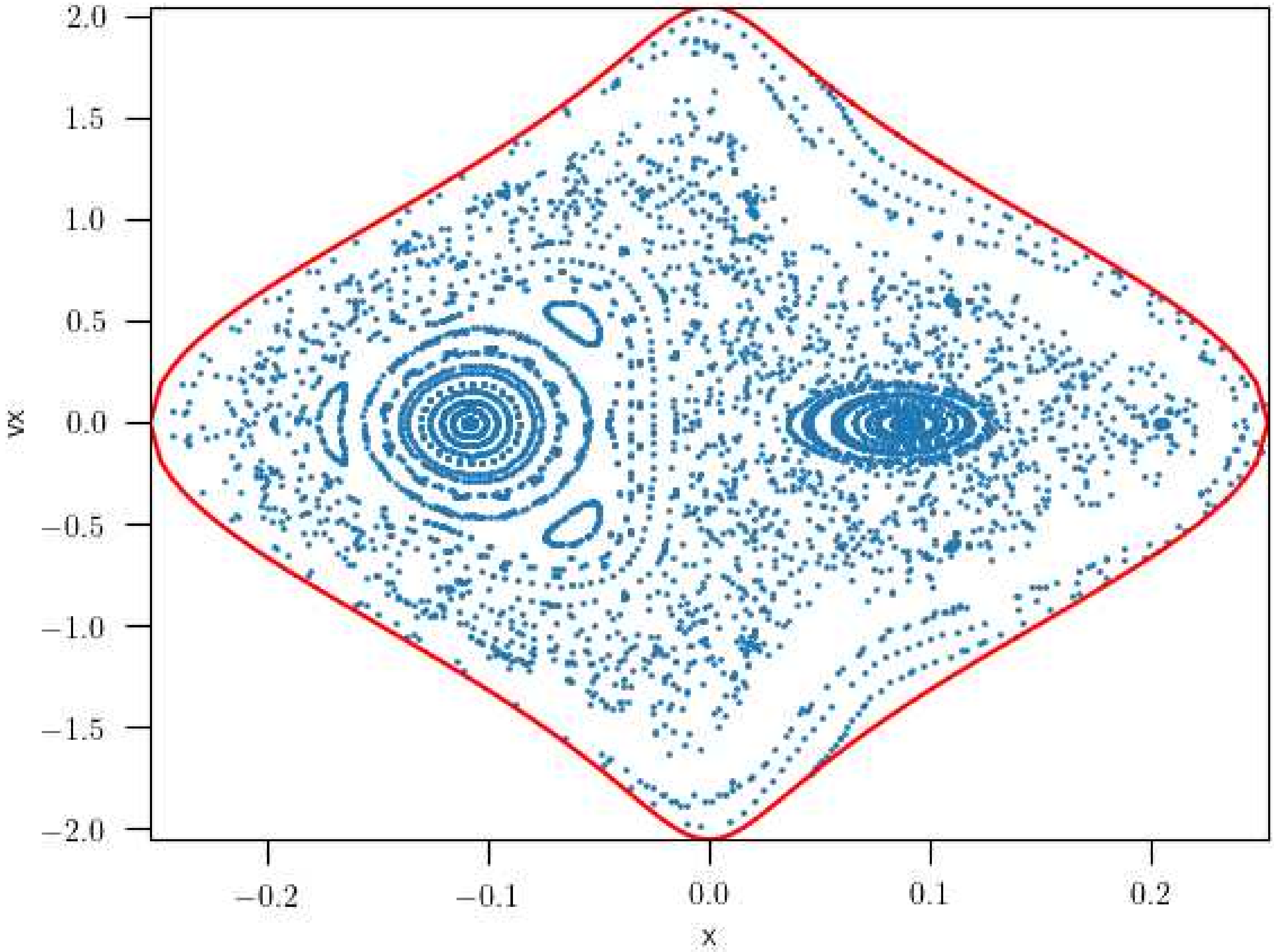

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0268
```

x1 : prograde x4 : retrograde



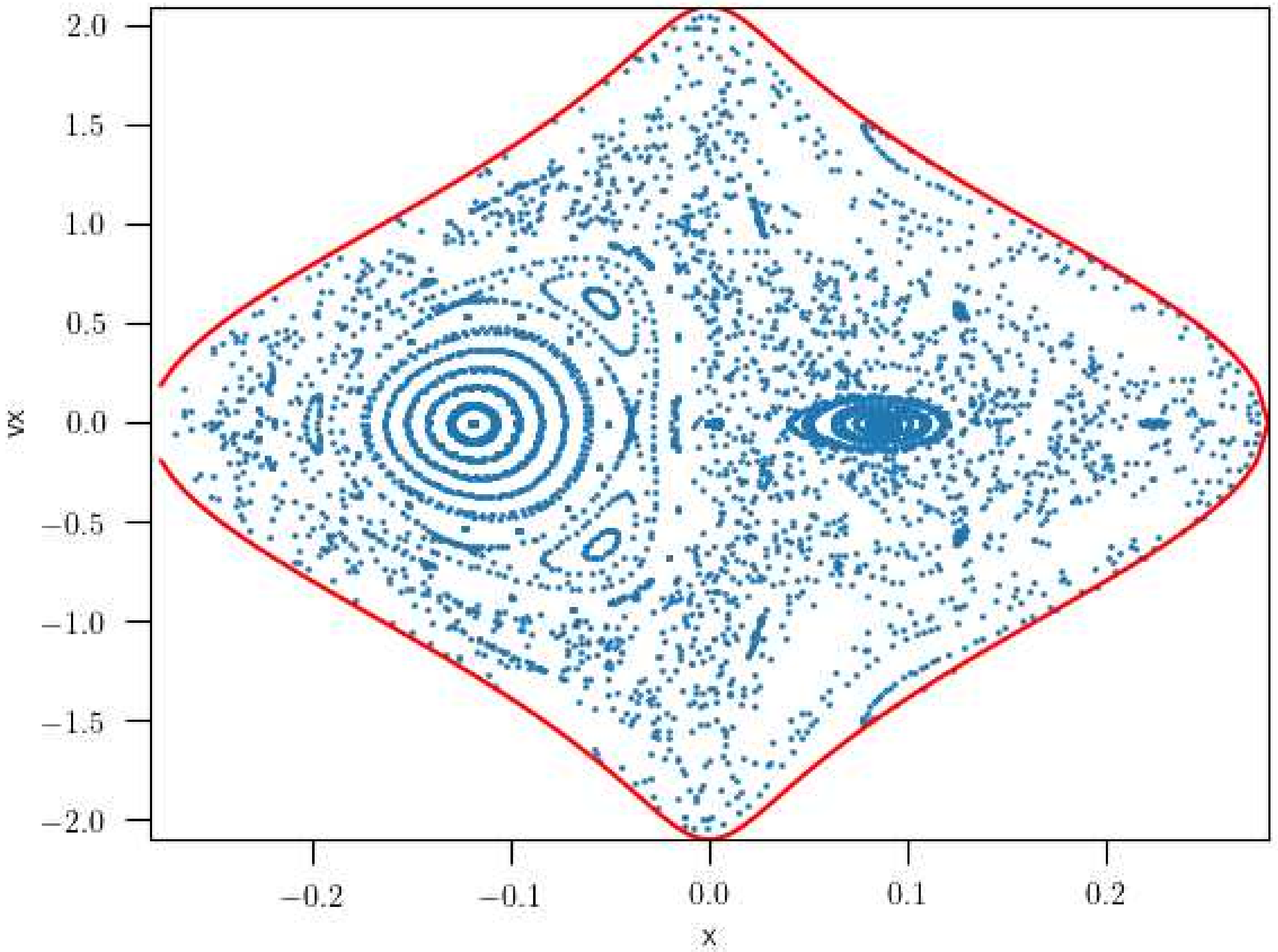

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x -0.034
```

$$E = -1.4$$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.4 --norbits 50
```

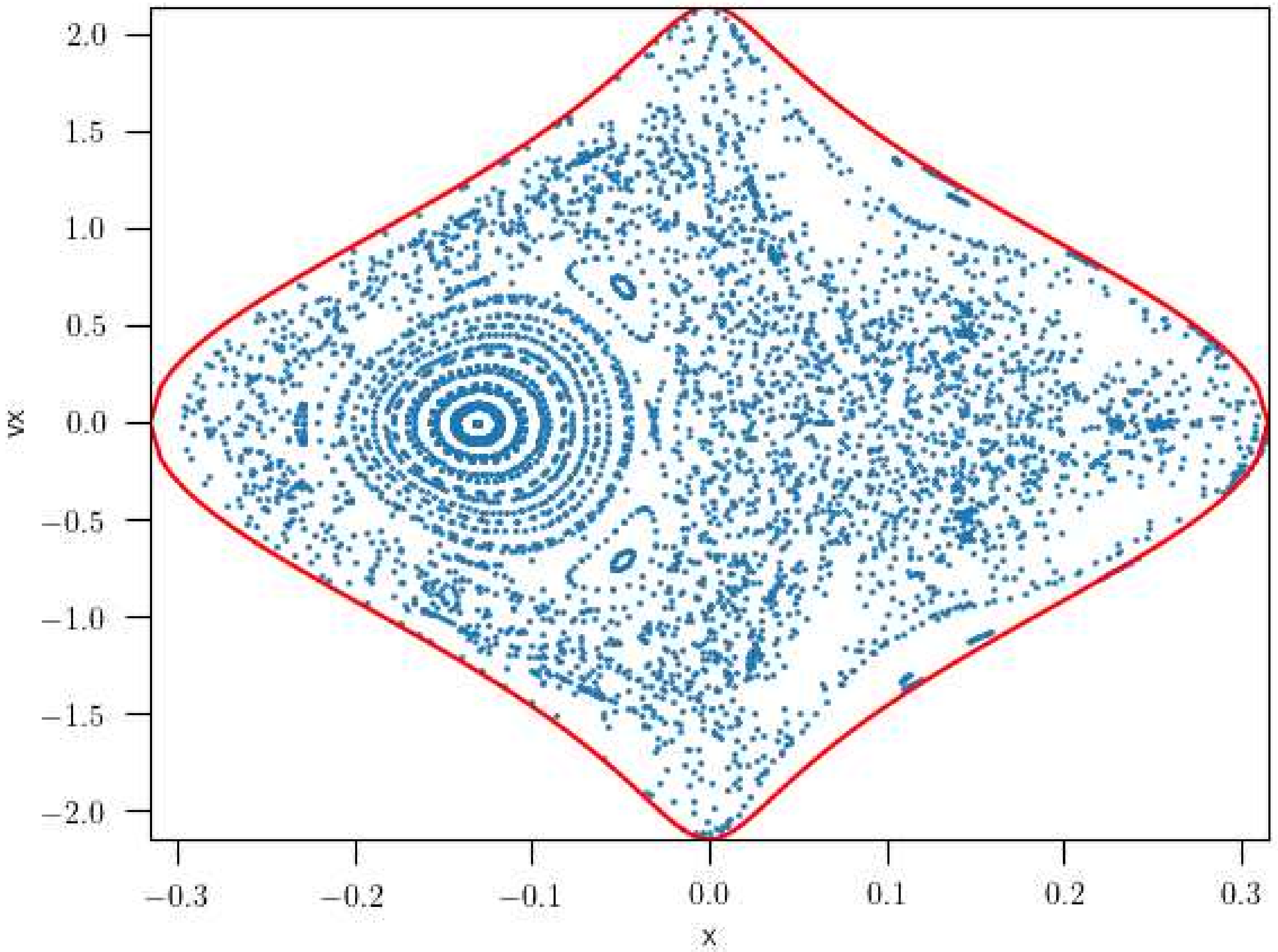
$$E = -1.3$$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.3 --norbits 50
```

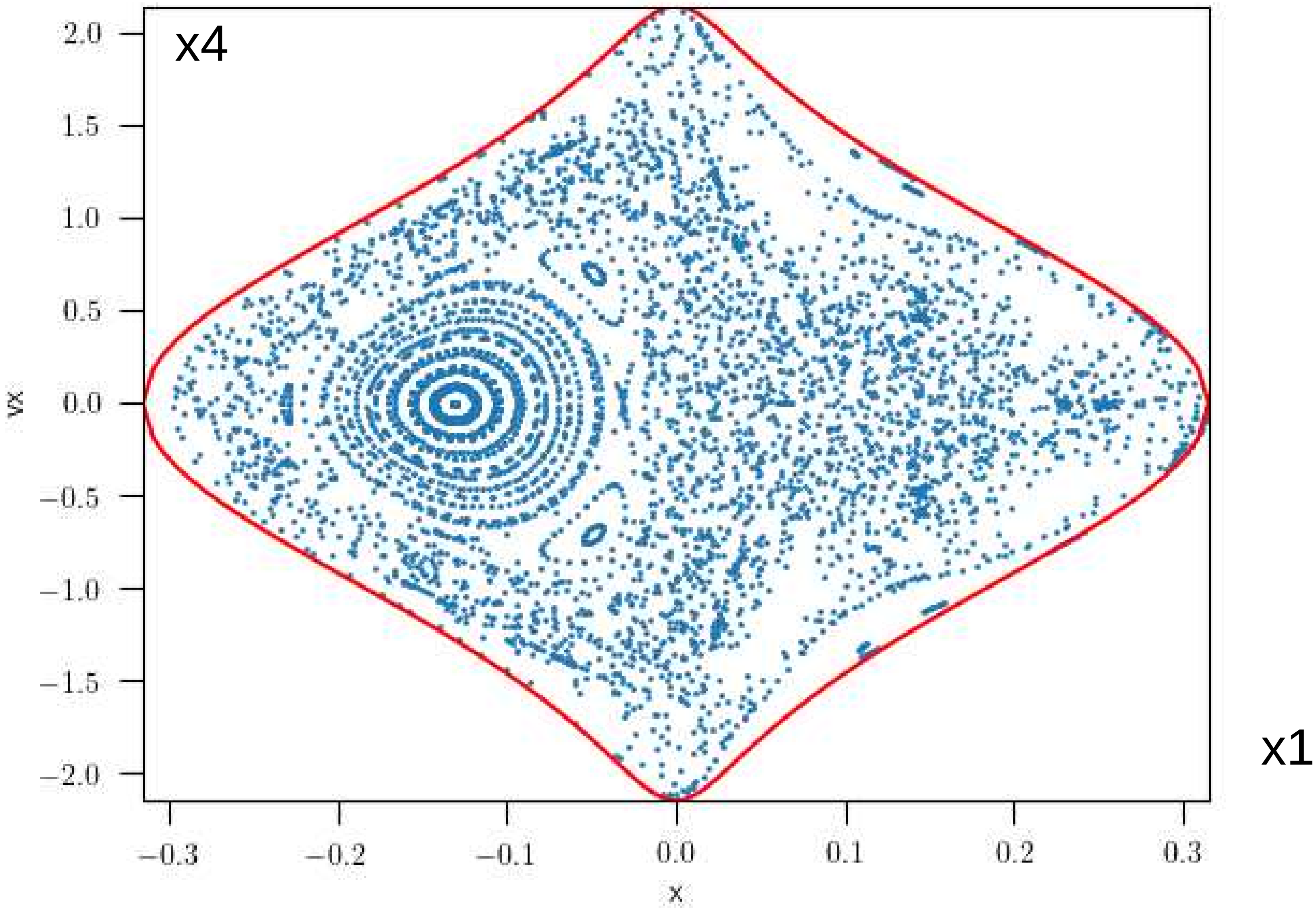
$E = -1.2$

Bifurcation : x2/x3 disappeared




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --norbits 50
```

$E = -1.2$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x -0.1283
```

The X-orbit families
(characteristics curves)

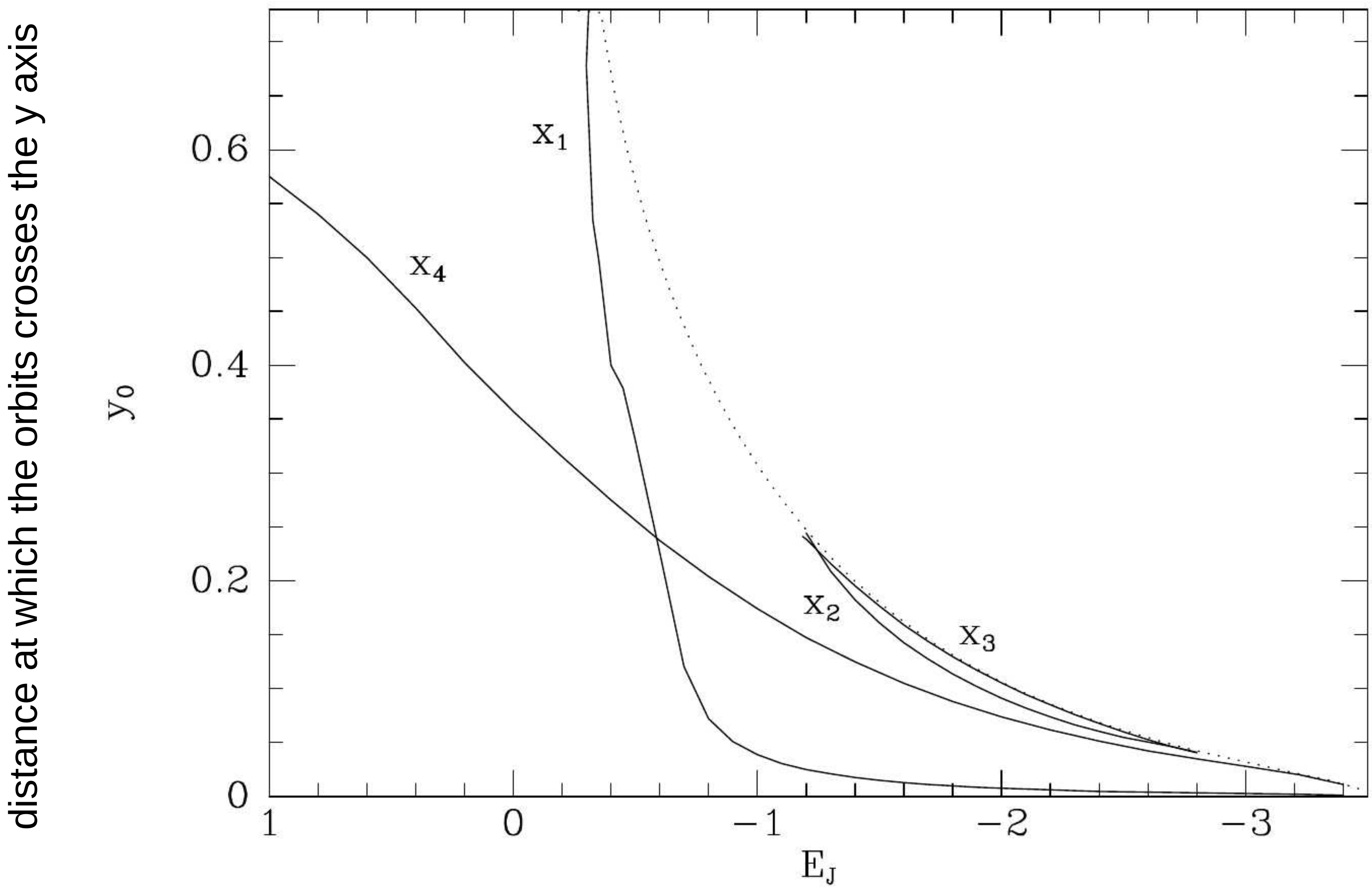


Figure 3.18 A plot of the Jacobi constant E_J of closed orbits in $\Phi_L(q = 0.8, R_c = 0.03, \Omega_b = 1)$ against the value of y at which the orbit cuts the potential's short axis. The dotted curve shows the relation $\Phi_{eff}(0, y) = E_J$. The families of orbits x_1–x_4 are marked.

Objective

- Split a loop orbit in two parts:

    - a circular motion of a guiding center

    - oscillations around the guiding center

Orbits in weak rotating bars (planar potentials)

- the barred potential rotates with a pattern speed Ω_b

Lagrangian :

$$\mathcal{L}(\vec{x},\dot{\vec{x}}) = \frac{1}{2}(\dot{\vec{x}} + \vec{\Omega}_b \times \vec{x})^2 - \phi(\vec{x})$$

In 2-D, with $\vec{\Omega}_b = \Omega_b \vec{e}_z$

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x} - y\Omega_b)^2 + \frac{1}{2}(\dot{y} + x\Omega_b)^2 - \phi(x,y)$$

In cylindrical coordinates

$$\boxed{\mathcal{L}(R, \varphi, \dot{R}, \dot{\varphi}) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}(R(\dot{\varphi} + \Omega_b))^2 - \phi(R,\varphi)}$$

Equations of motion in cylindrical coordinates (Euler-Lagrange)

$$\begin{cases} \ddot{R} = R(\dot{\varphi} + \Omega_b)^2 - \dfrac{\partial \phi}{\partial R} \\ \dfrac{d}{dt}\left(R^2(\dot{\varphi} + \Omega_b) \right) = - \dfrac{\partial \phi}{\partial \varphi} \end{cases}$$

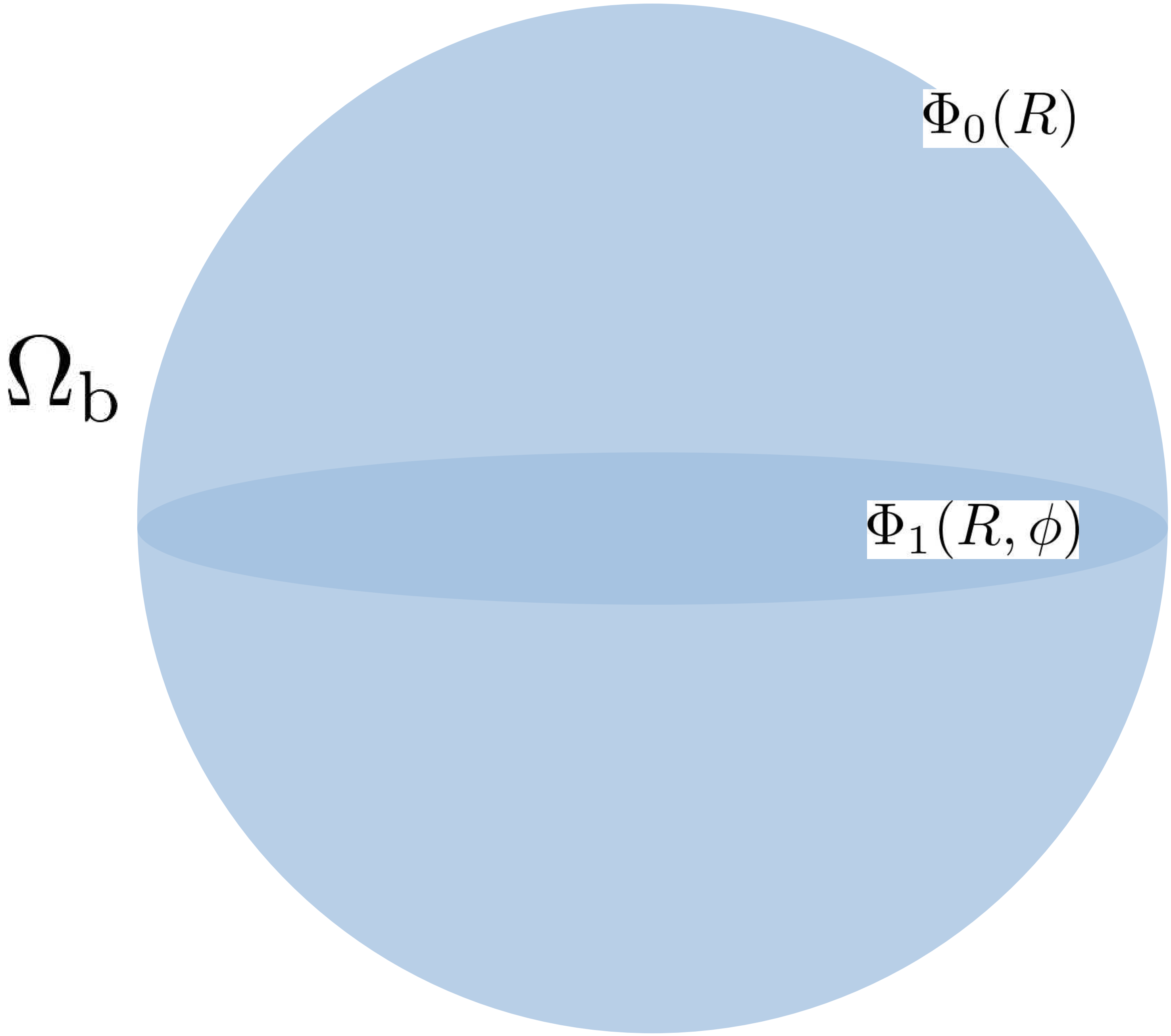
Assumptions

(1) A weak perturbation : $\phi(R,\varphi) = \underbrace{\phi_0(R)}_{\text{cyl. symmetry}} + \underbrace{\phi_1(R,\varphi)}_{\text{perturbation}}$ $\quad \dfrac{|\phi_1|}{|\phi_0|} \ll 1$

$$\phi_1(R,\varphi) = \underbrace{\phi_b(R)}_{\text{radial dependency}} \underbrace{\cos(m\varphi)}_{\text{azimuthal dependency}}$$

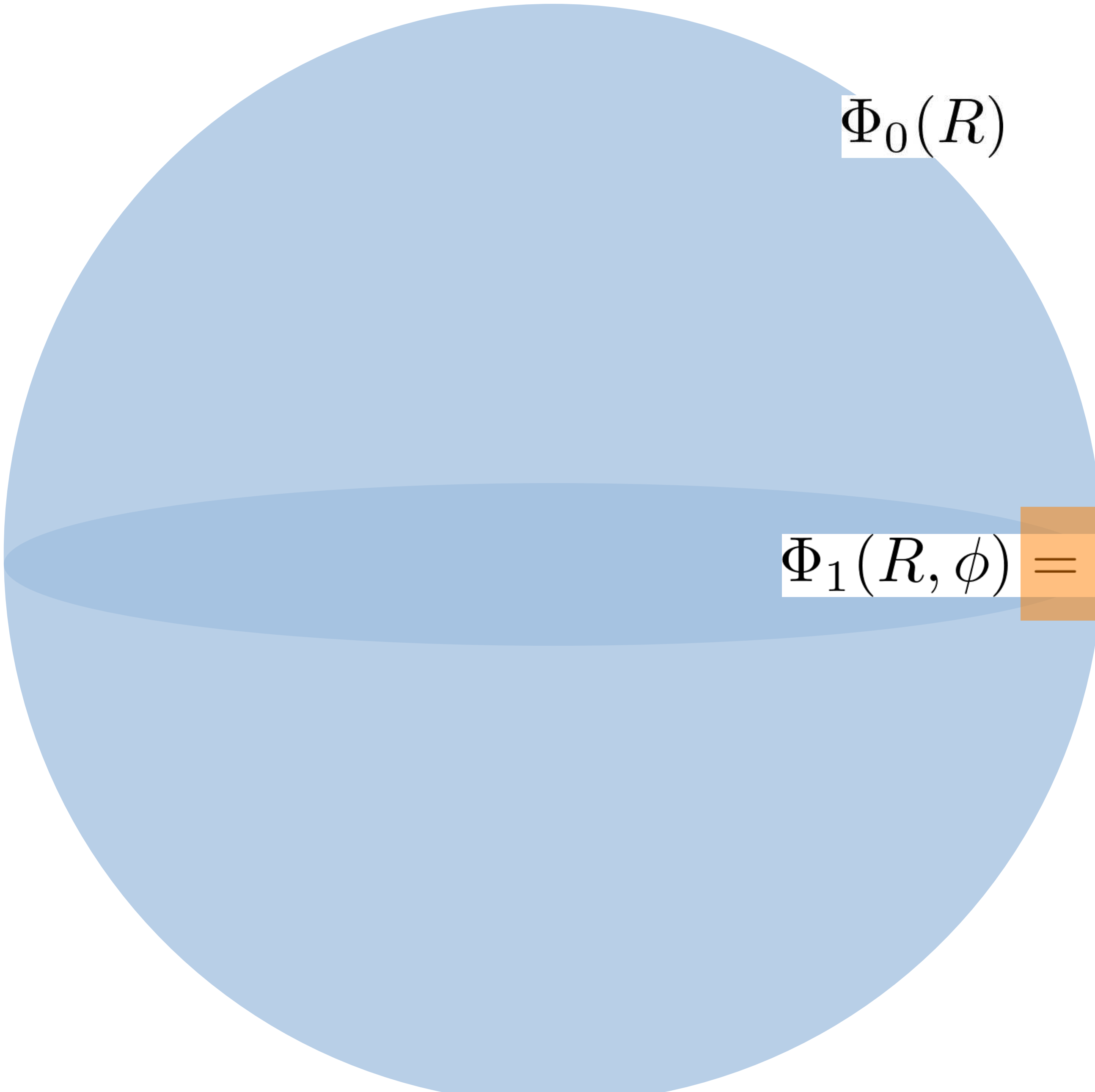
m : perturbation mode

The weakly-bared galaxy model



$$\Phi_1 \ll \Phi_0$$

The weakly-bared galaxy model



$$\Omega_\mathrm{b}$$

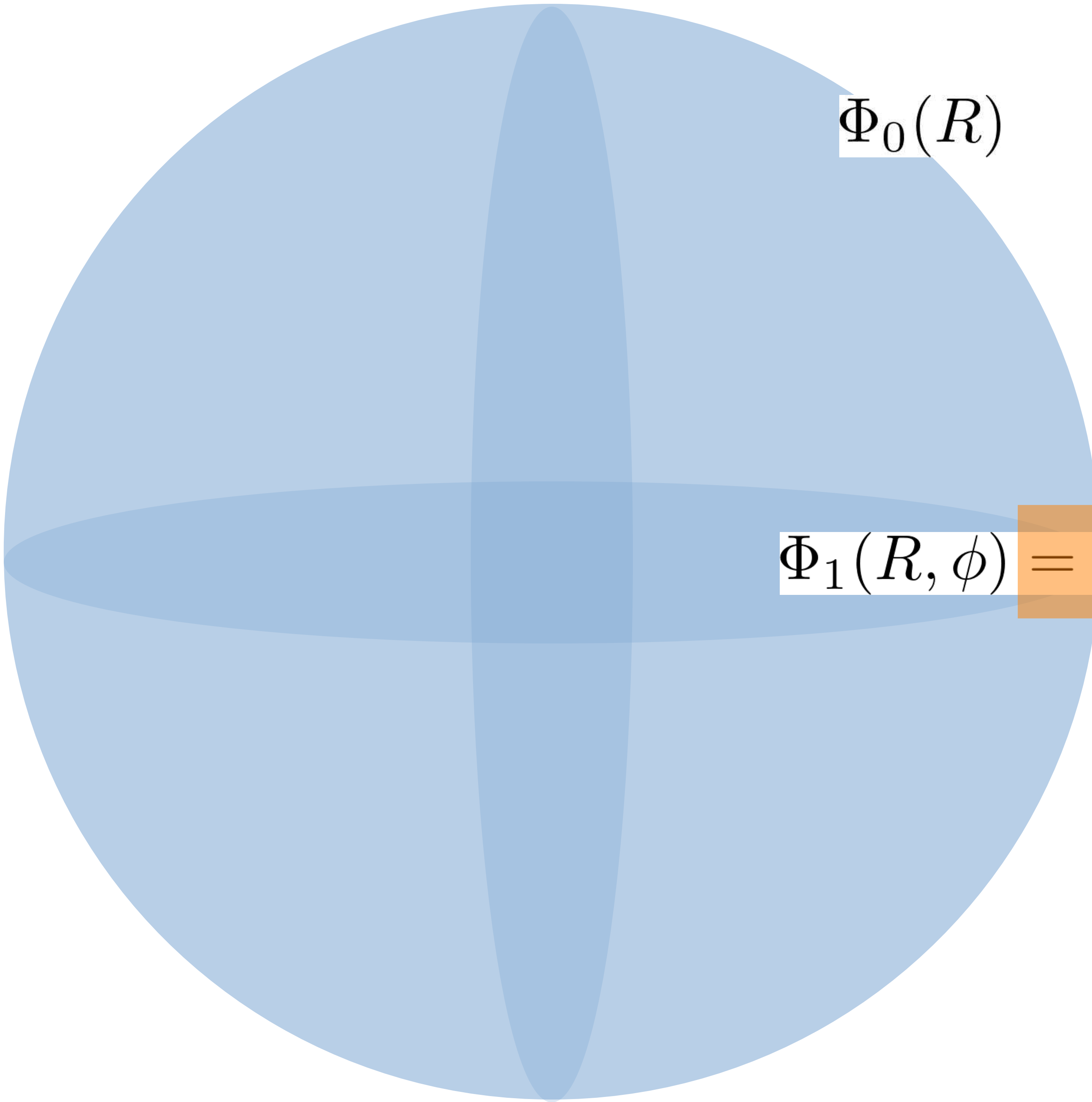
$$\Phi_0(R)$$

$$\Phi_1(R,\phi) = \Phi_\mathrm{b}(R)\,\cos(m\phi)$$

$$m = 2$$

$$\Phi_1 \ll \Phi_0$$

The weakly-bared galaxy model



$$\Omega_{\mathrm{b}}$$

$$\Phi_0(R)$$

$$\Phi_1(R, \phi) = \Phi_{\mathrm{b}}(R)\, \cos(m\phi)$$

$$m = 4$$

$$\Phi_1 \ll \Phi_0$$

Assumptions

② The motion may be decomposed into two parts

1) circular motion
2) perturbation

$$\begin{cases} R(t) = R_0(t) + R_1(t) \\ \varphi(t) = \varphi_0(t) + \varphi_1(t) \end{cases}$$

$R_1 \ll R_0$

$\varphi_1 \ll \varphi_0$

Note

$$\begin{cases} R_0(t) = R_0 & (R_0 = \text{radius of the guiding center}) \\ \varphi_0(t) = (\Omega_0 - \Omega_b)\, t & (\Omega_0 = \text{circular frequency}) \end{cases}$$

Solution of the EoM (2nd order terms)

EXERCICE

Radial motion

$$R_1(\varphi_0) = C_1 \cos\left(\frac{\kappa_0 \varphi_0}{\Omega_0-\Omega_b} + \alpha\right) - \left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega-\Omega_b)}\right]_{R_0} \frac{\cos(m\varphi_0)}{\kappa_0^2 - m^2(\Omega_0-\Omega_b)^2}$$

C_1, α : arbitrary constants

κ_0 : radial epicycle frequency

Azimuthal motion

$$\dot{\varphi}_1(t) = -2\Omega_0 \frac{R_1}{R_0} - \frac{\Phi_b(R_0)}{R_0^2(\Omega_0-\Omega_b)}\cos\left(m(\Omega_0-\Omega_b)t\right) + cte$$

Discussion

$$R_1(\varphi_0) = C_1 \cos\left(\frac{\varkappa_0 \varphi_0}{\Omega_0-\Omega_b} + \alpha\right) - \left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega-\Omega_b)}\right]_{R_0} \frac{\cos(m\varphi_0)}{\varkappa_0^2 - m^2(\Omega_0-\Omega_b)^2}$$

$\varphi_0 = (\Omega_0 - \Omega_b)\,t$

① if $\Phi_b(R) = 0$ (no perturbation)

Epicycles motions

$$R_1(t) = C_1 \cos(\varkappa_0 t + \alpha)$$ $\equiv x(t)$ radial oscillations

$$\dot{\varphi}_1(t) = -2\Omega_0 \frac{R_1(t)}{R_0}$$ $\Rightarrow y(t)$ oscillations along the orbit

② if $C_1 = 0$, $\Phi_b \neq 0$

$$R_1(\varphi_0) = -\left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega-\Omega_b)}\right]_{R_0} \frac{\cos(m\varphi_0)}{\varkappa_0^2 - m^2(\Omega_0-\Omega_b)^2}$$

cte — periodic in φ_0 $\left(\frac{2\pi}{m}\right)$

$\Rightarrow$ closed orbit

③ if $C_1 \neq 0$ oscillations around the closed orbit (same family)

The orbit is not necessary closed

Resonances

! two problematic terms $\frac{1}{\Omega_0 - \Omega_b}$ and $\frac{1}{\varkappa_0^2 - m^2(\Omega_0 - \Omega_b)^2}$

$\Rightarrow$ R_1 may diverge !

1) $\boxed{\Omega_0 = \Omega_b}$ Corotation

we are at a radius where the circular frequency is similar to the pattern speed of the bar

as $\dot{\varphi}_0 = \Omega_0 - \Omega_b \Rightarrow \dot{\varphi}_0 = 0$ → static in the rotating frame

2) $\boxed{m(\Omega_0 - \Omega_b) = \pm \varkappa_0}$ Lindblad resonances

freqr. at which the star encounter the potential minimum

→ the frequency at wich a star encounter a potential minimum is similar to its radial frequency

$\Rightarrow$ exitation

$\equiv \Omega_b = \Omega \pm \frac{\varkappa}{2}$

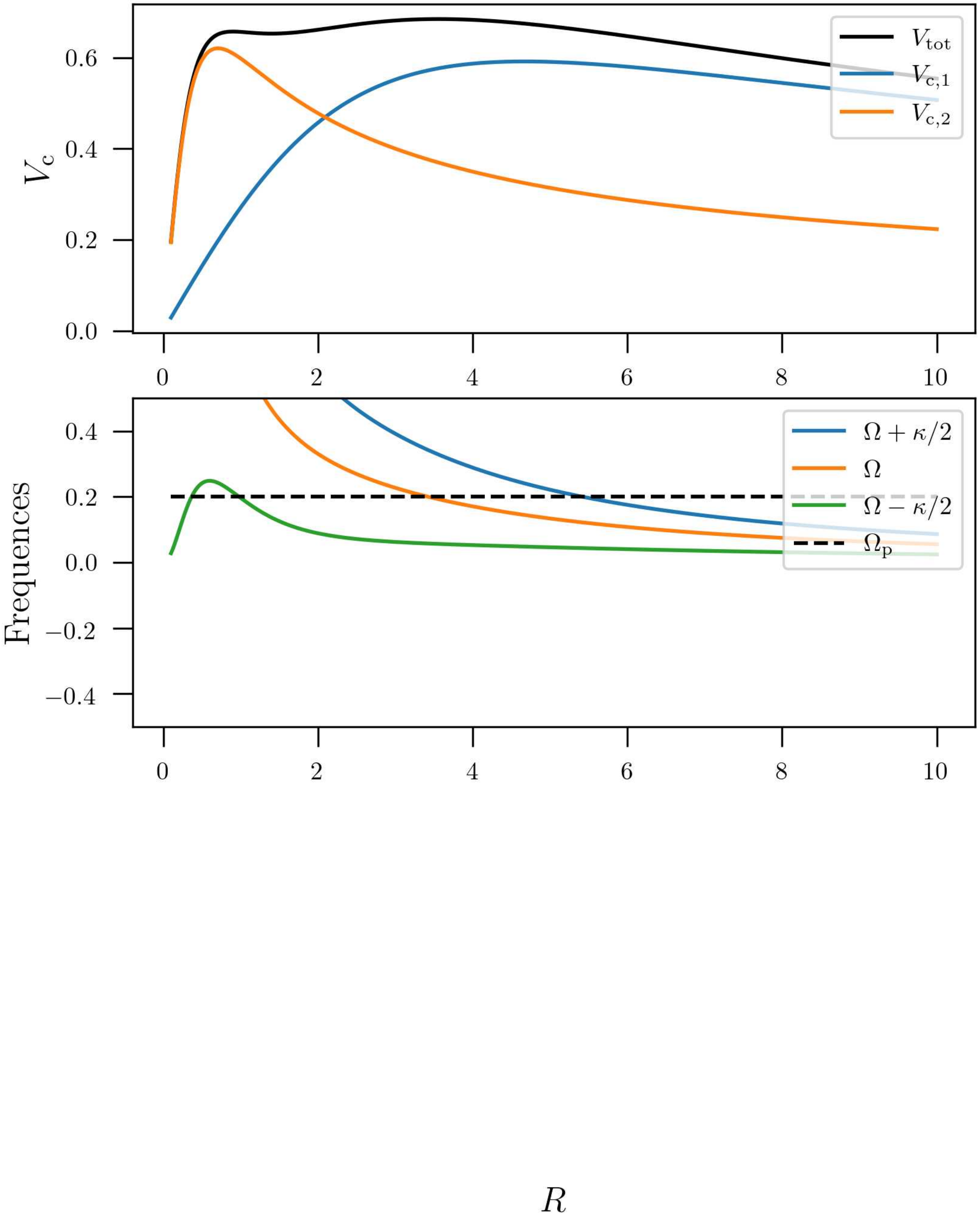
A circular orbit has two natural frequencies

① $\varkappa$: radial frequ.

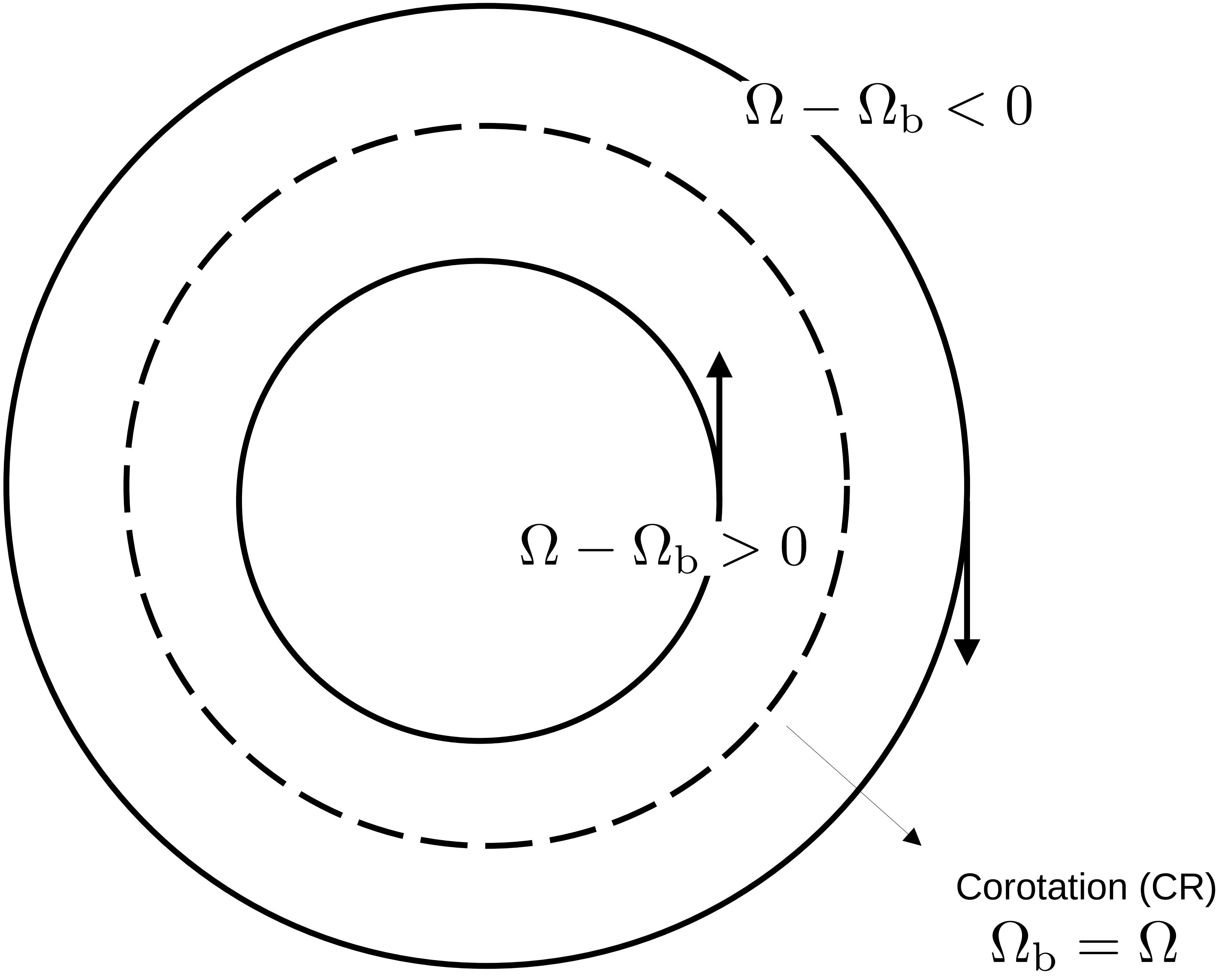
② 0 : azimuthal frequ. (no change $\Rightarrow$ frequ = 0)

Resonances occur when the forcing frequency $m(\Omega_0 - \Omega_b)$ is equal to one of these frequencies.

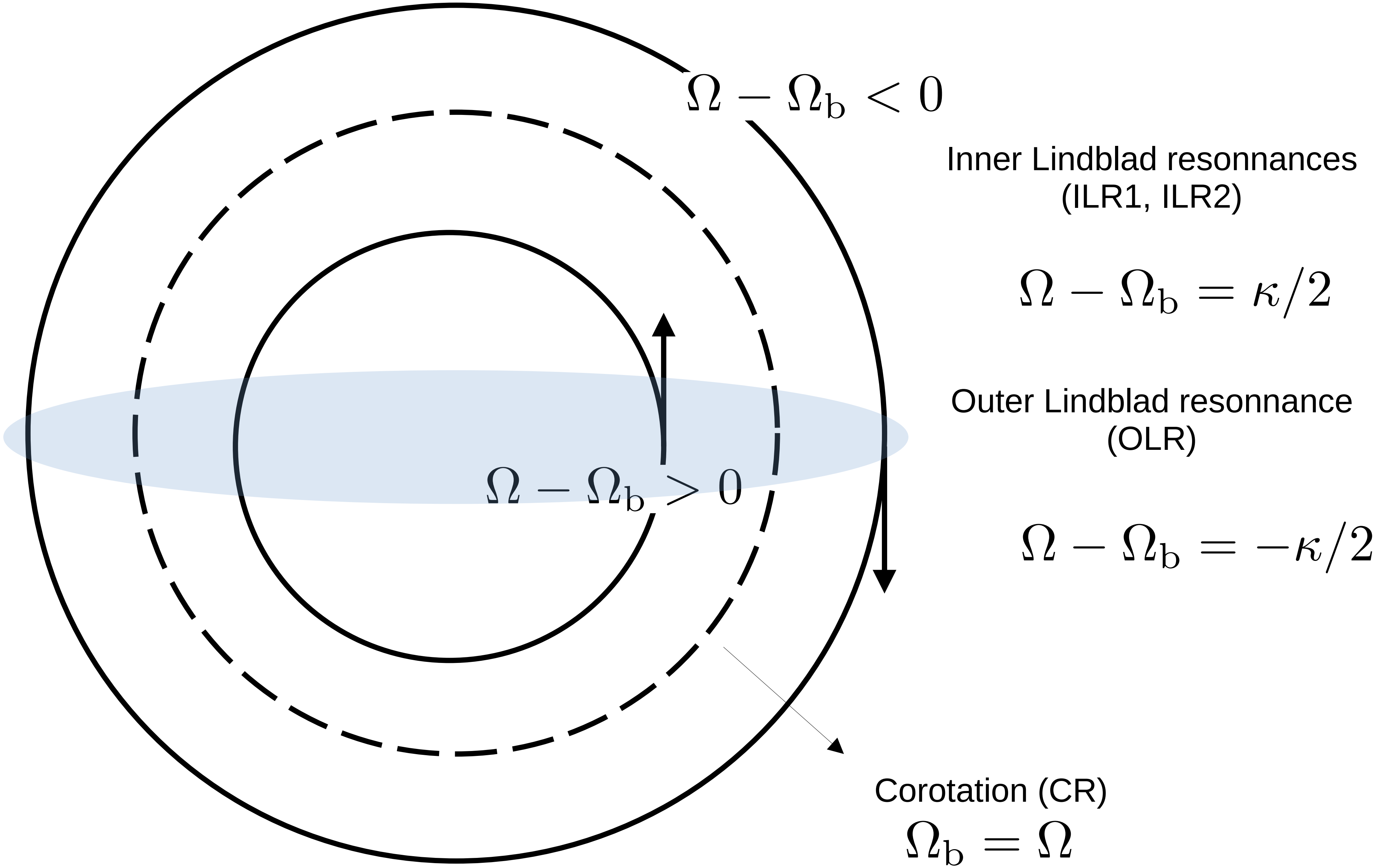
Disk : Miyamoto-Nagai
Bulge : Plummer

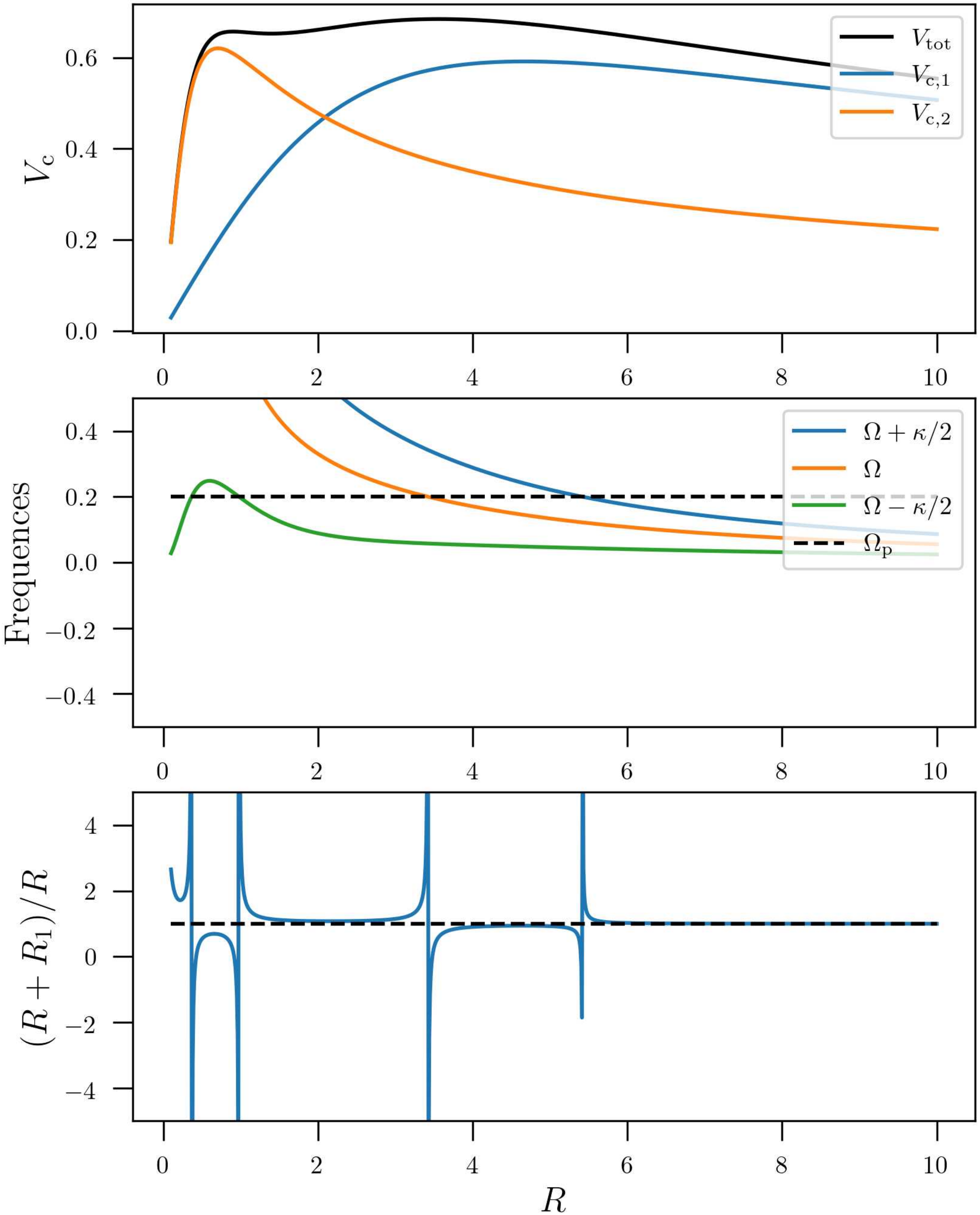


$\Omega = \Omega(R)$ is a decreasing function in a galaxy



$\Omega = \Omega(R)$ is a decreasing function in a galaxy





Disk : Miyamoto-Nagai
Bulge : Plummer

Inner Lindblad resonnances
(ILR1, ILR2)

$$\Omega_b = \Omega - \kappa/2$$

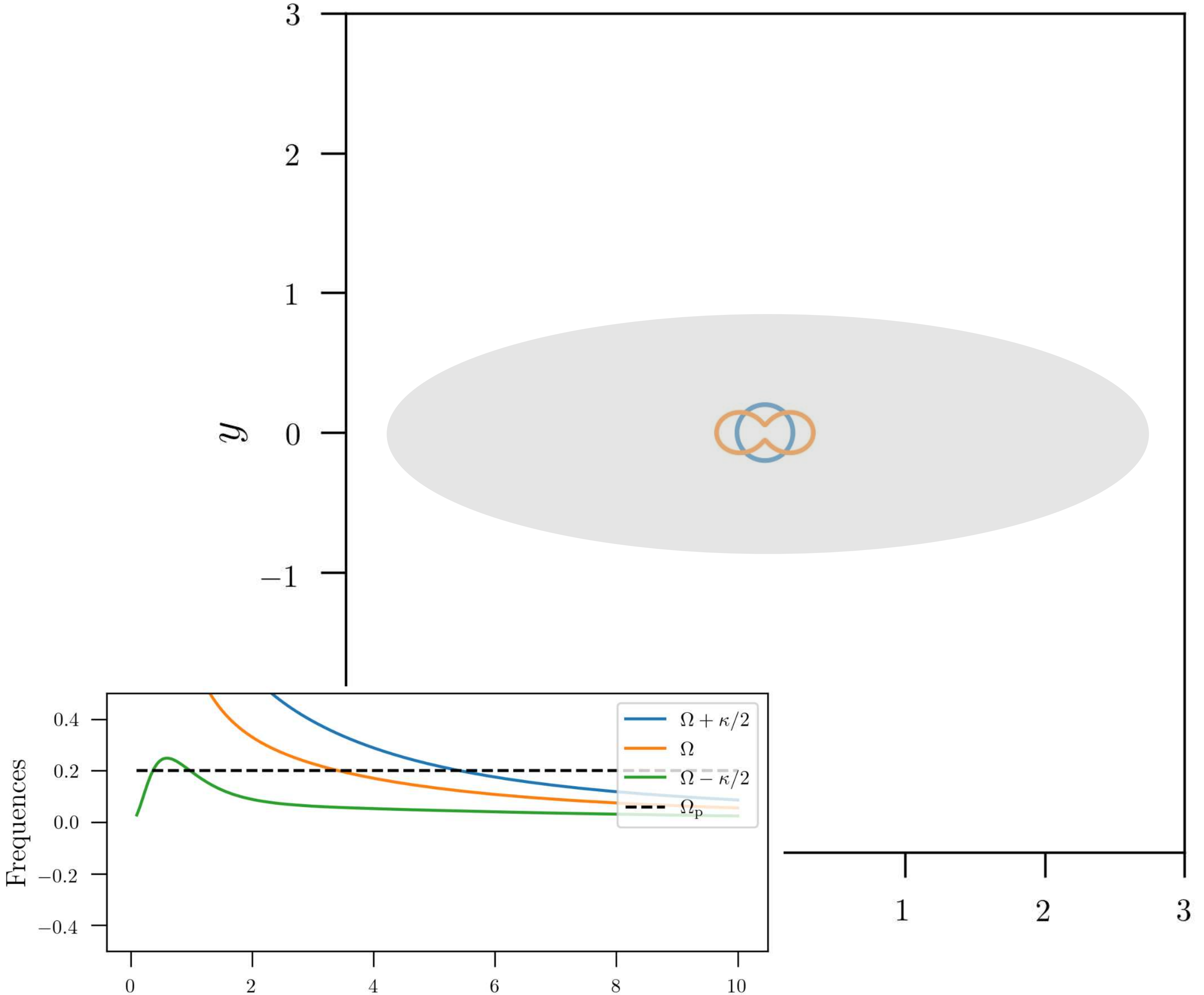
Outer Lindblad resonnance
(OLR)

$$\Omega_b = \Omega + \kappa/2$$

Corotation (CR)

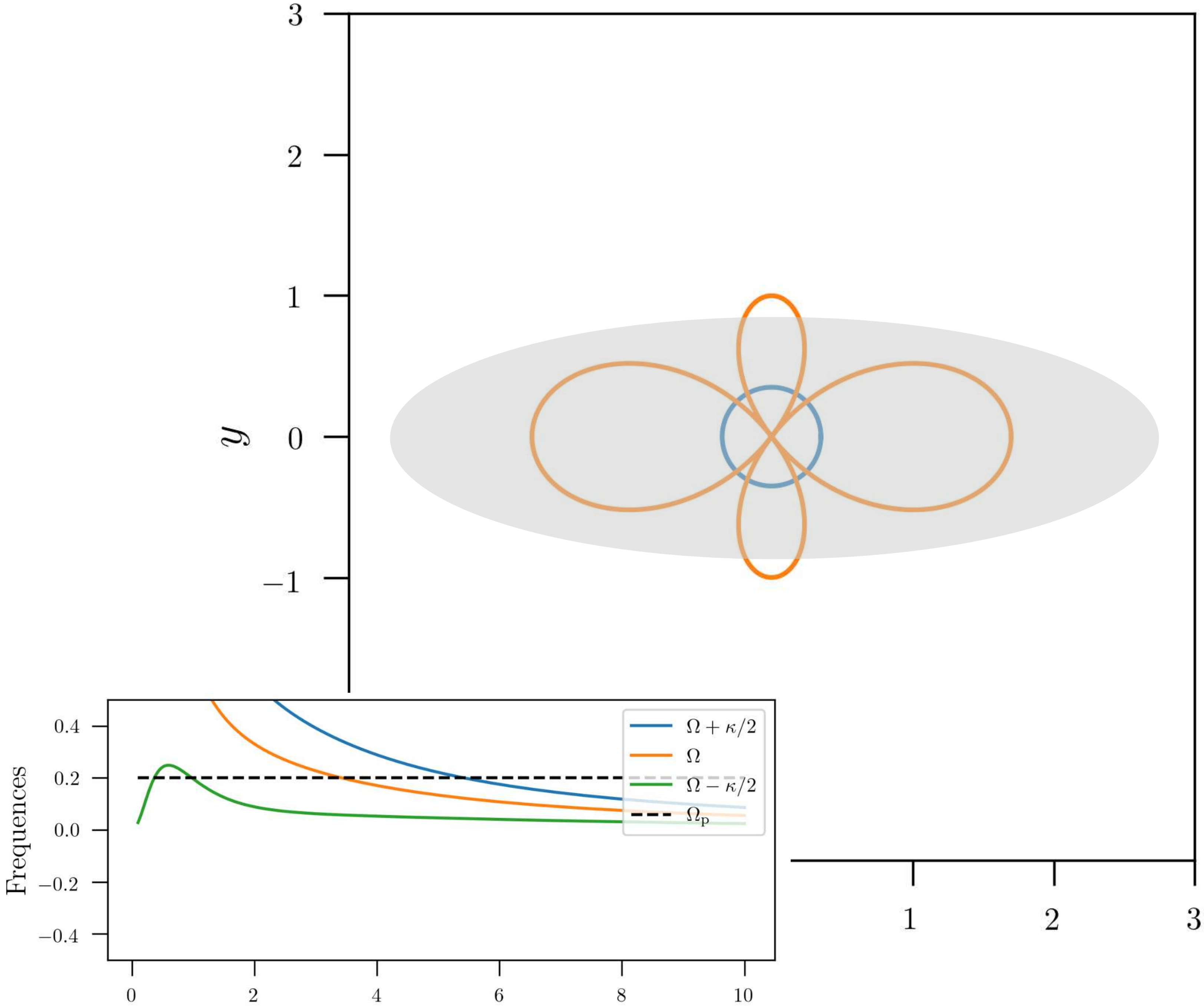
$$\Omega_b = \Omega$$

$R = 0.2$ $R < R_{\mathrm{ILR1}}$

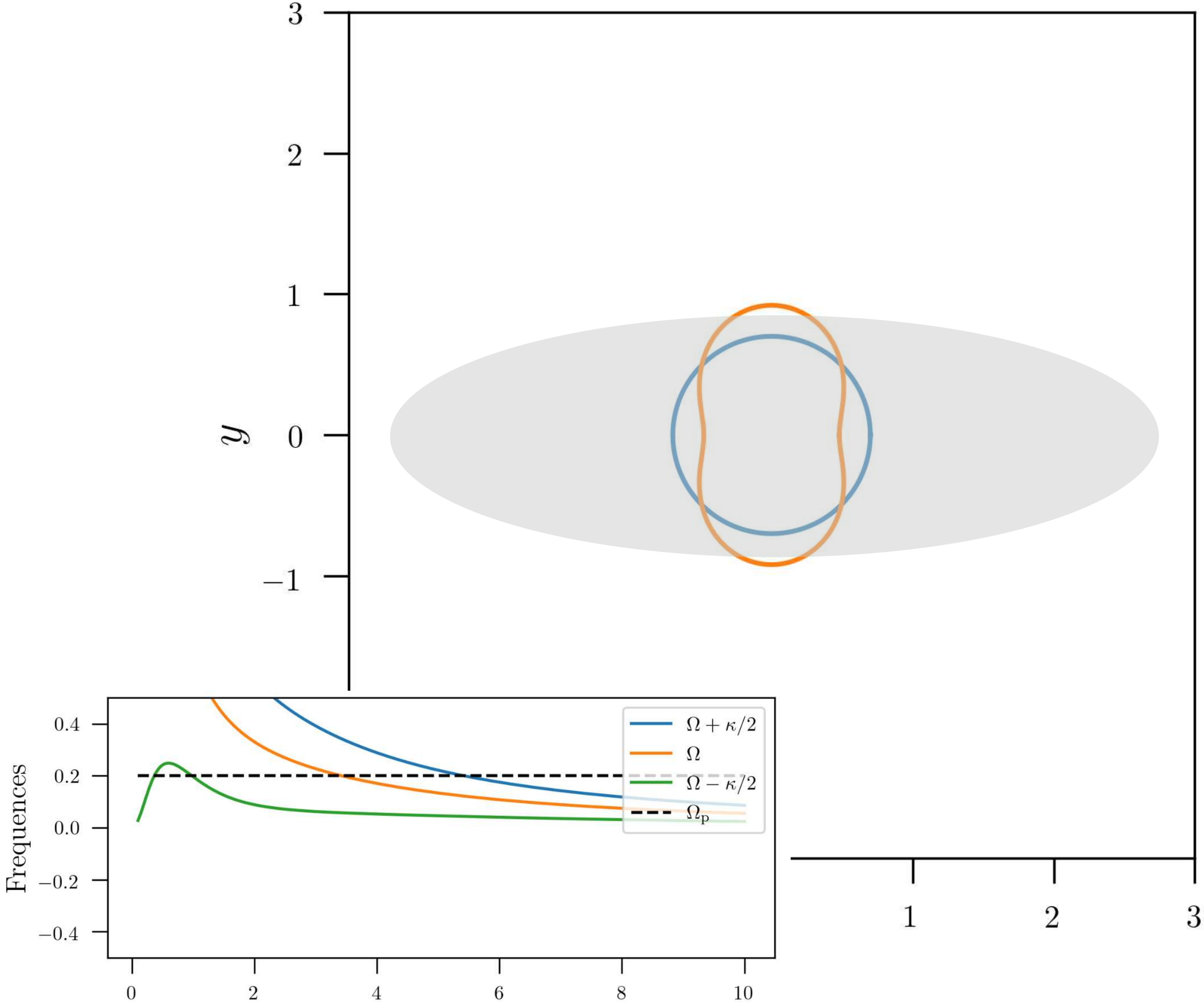


$R = 0.3$

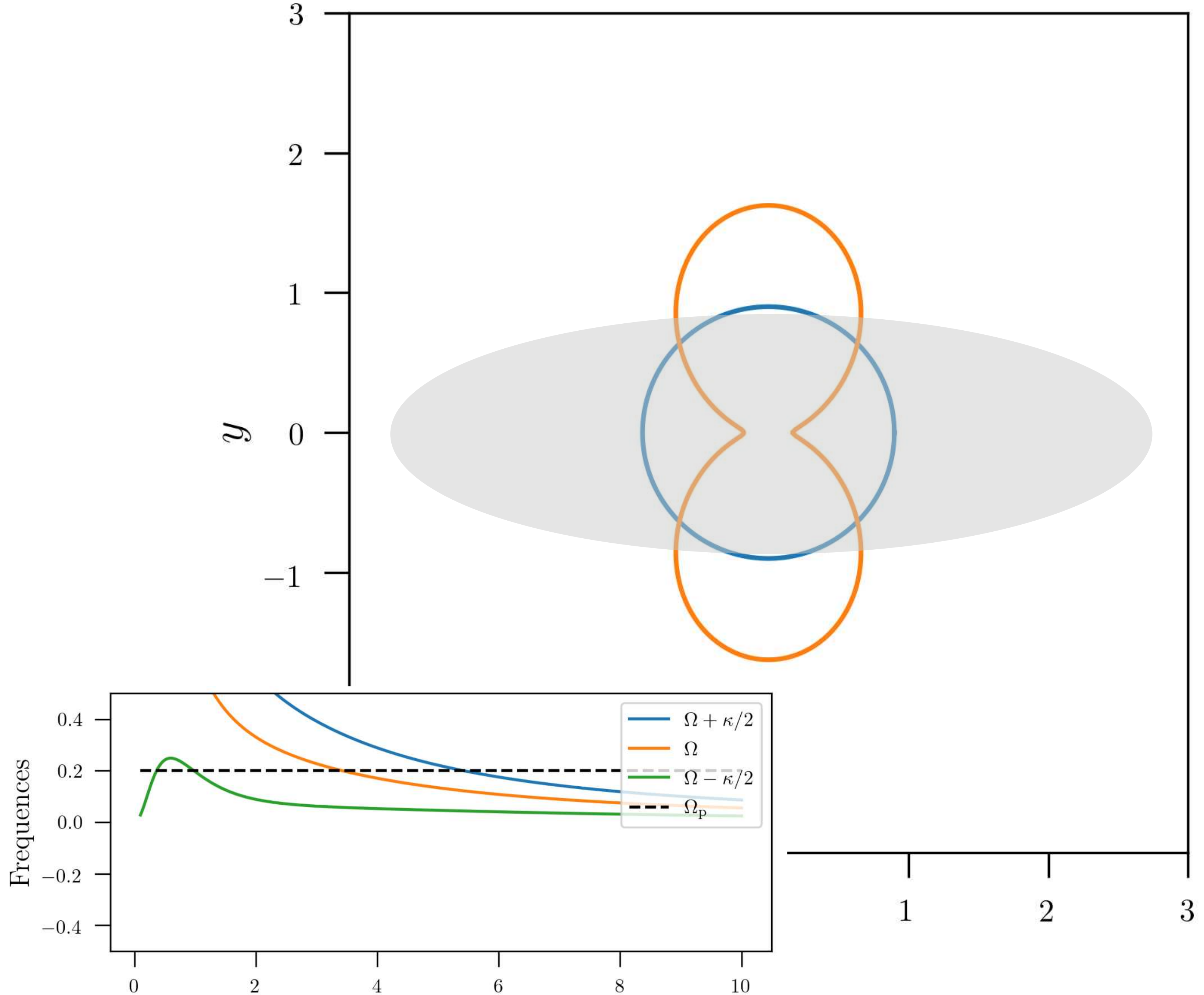
$R \cong R_{\mathrm{ILR1}}$



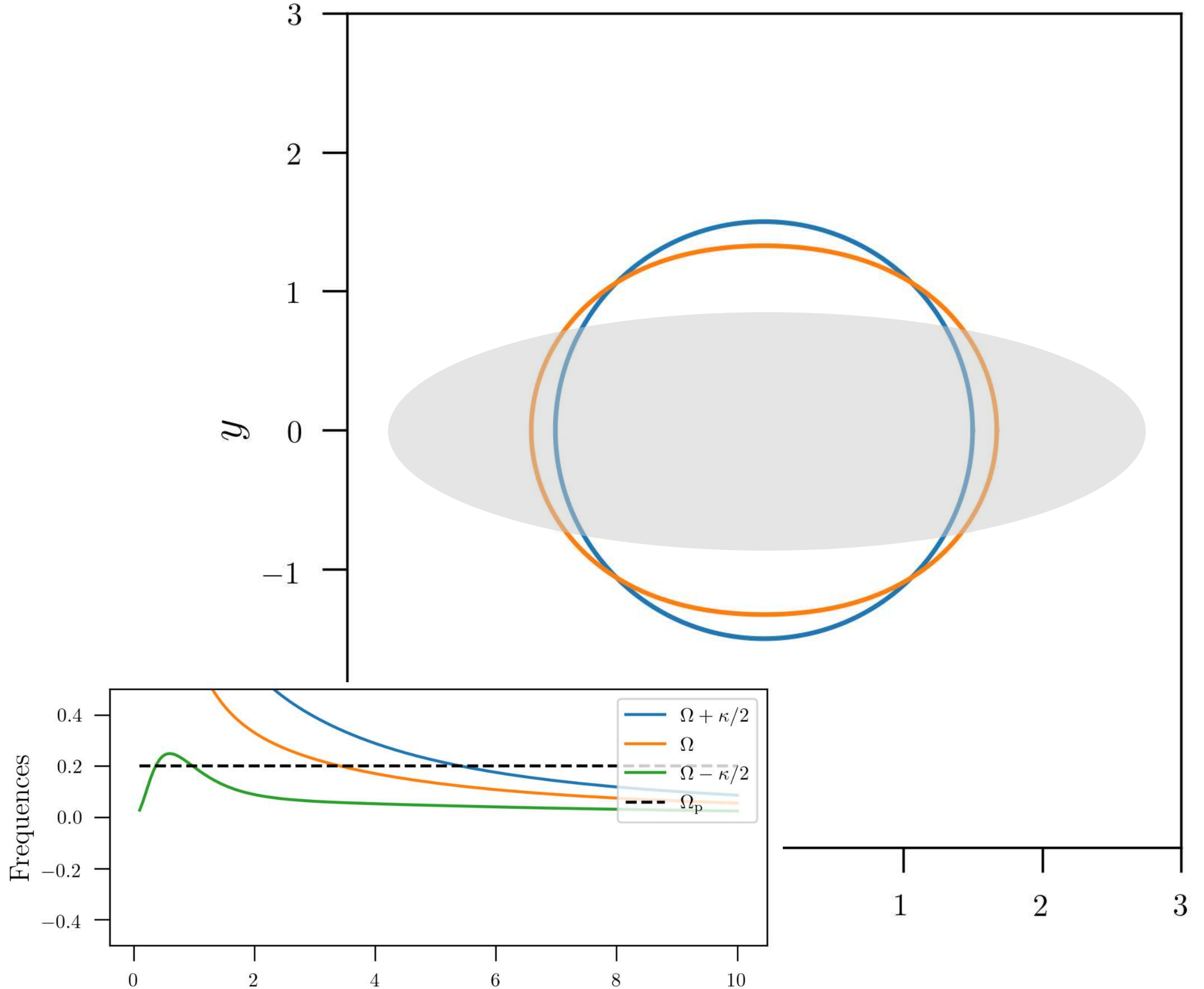
$R = 0.7$ $\qquad R_{\mathrm{ILR1}} < R < R_{\mathrm{ILR2}}$



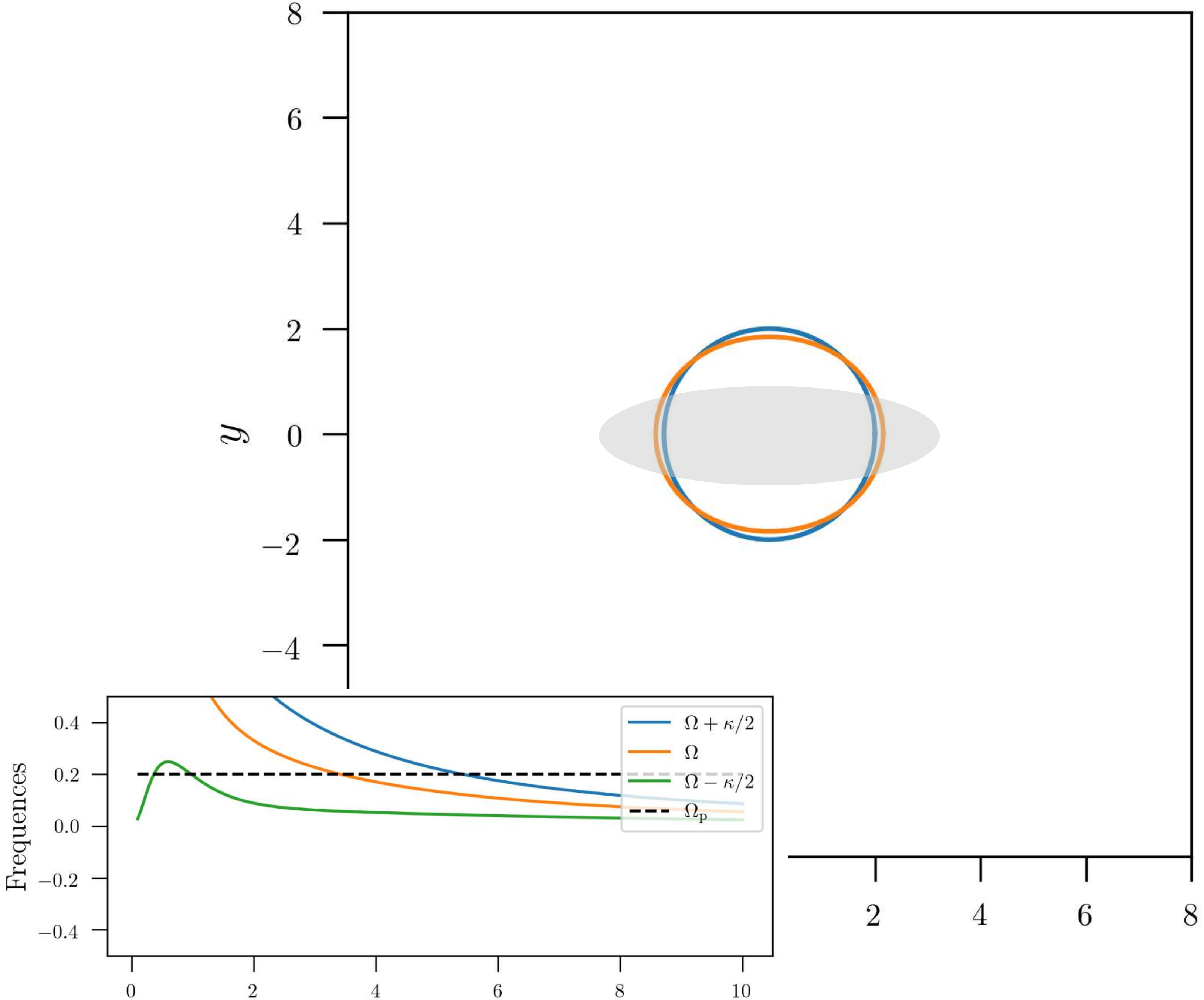
$R = 0.9$ $R \cong R_{\mathrm{ILR2}}$



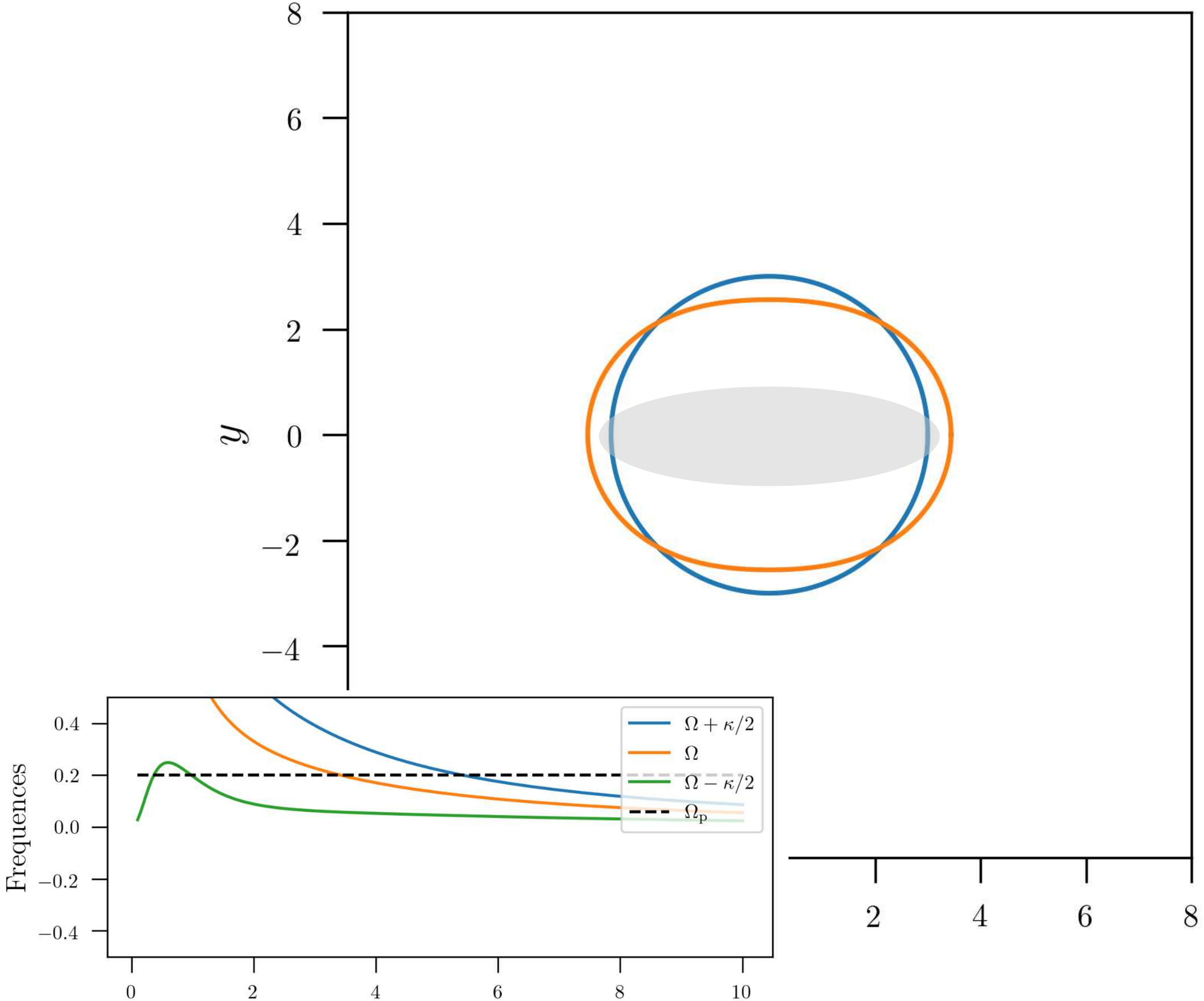
$R = 1.5$ $R_{\mathrm{ILR2}} < R < R_{\mathrm{CR}}$



$R = 2.0$ $\qquad$ $R_{\mathrm{ILR2}} < R < R_{\mathrm{CR}}$

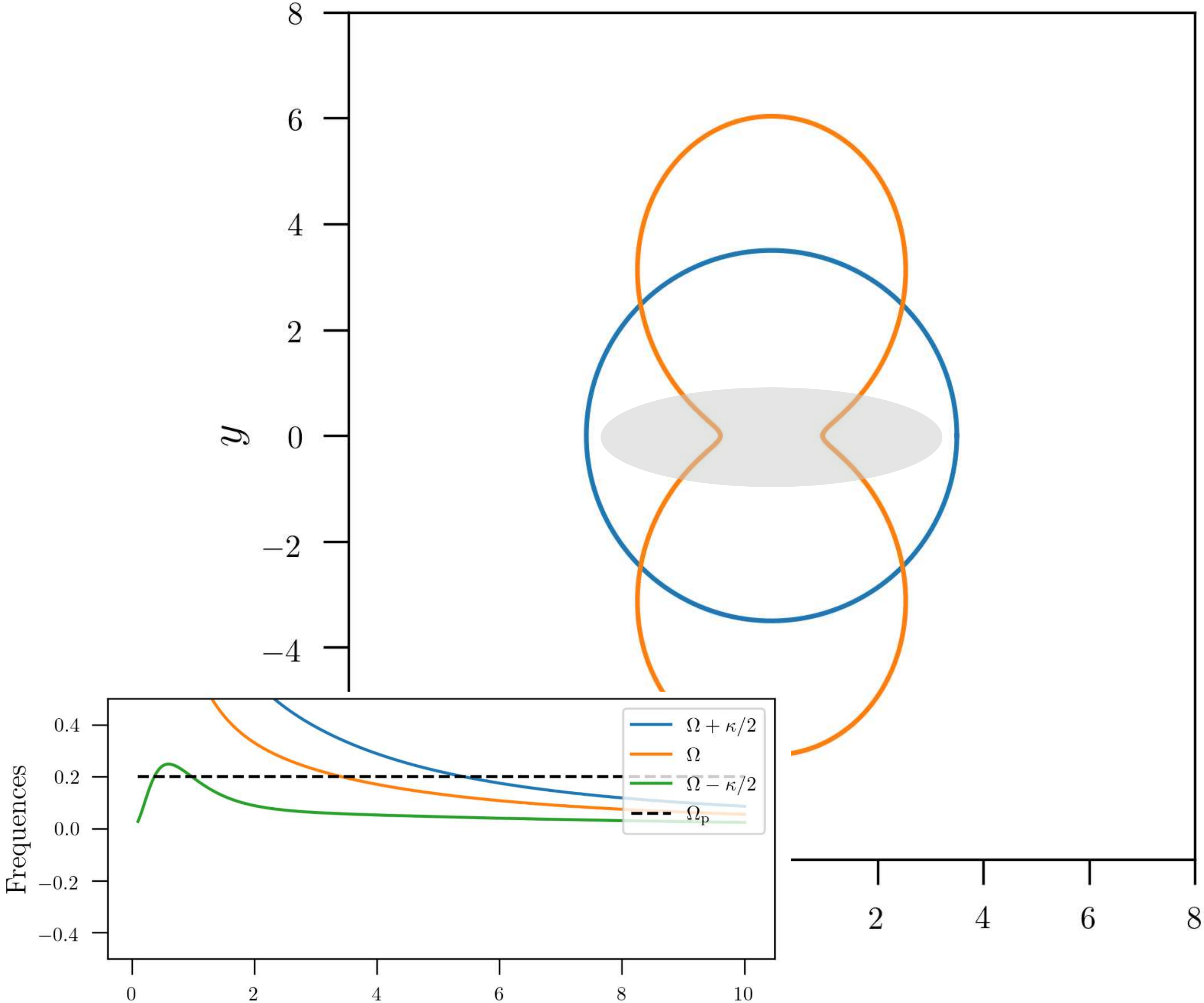


$R = 3.0$ $\quad$ $R_{\mathrm{ILR2}} < R < R_{\mathrm{CR}}$



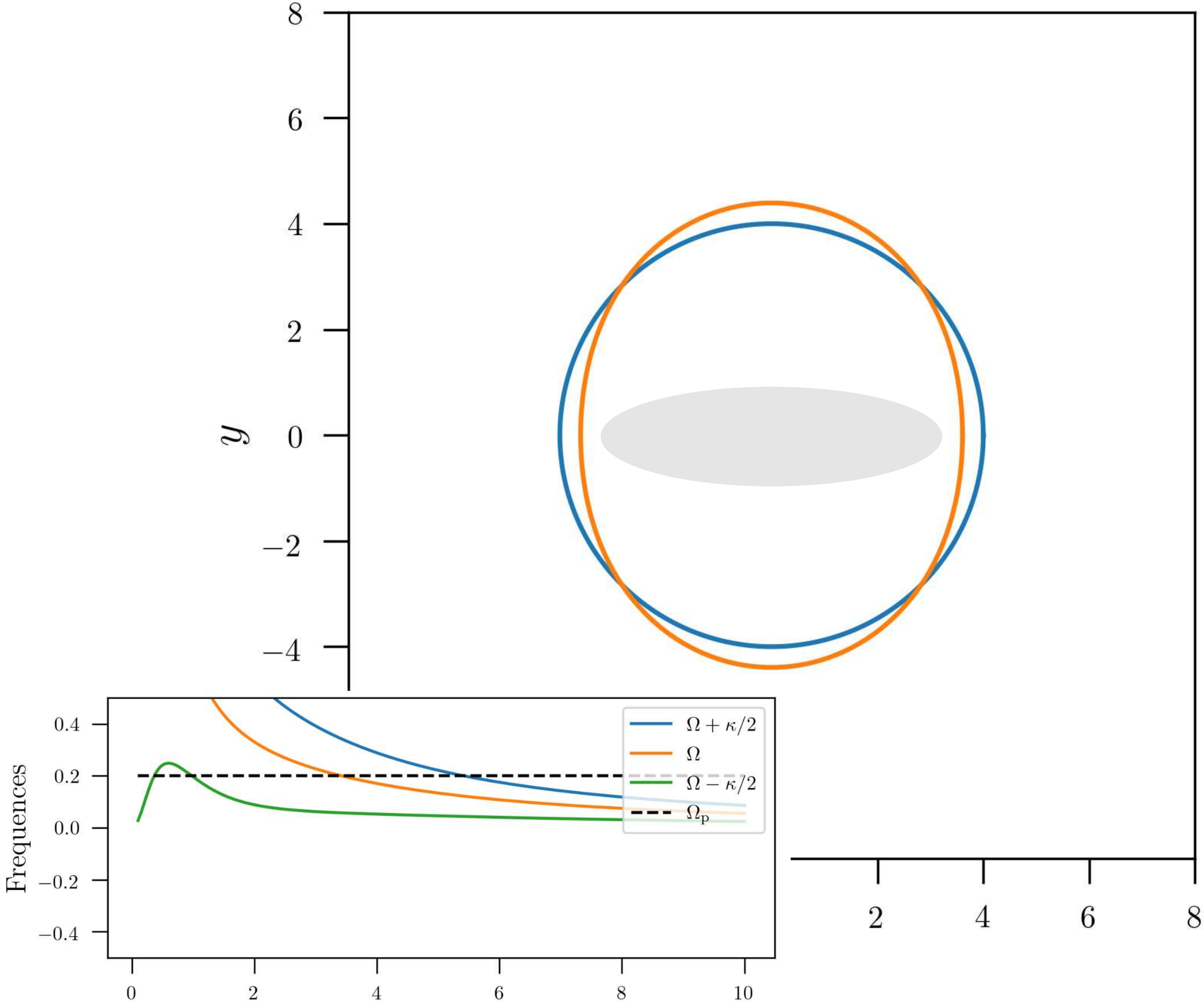
$R = 3.5$

$R \cong R_{\mathrm{CR}}$

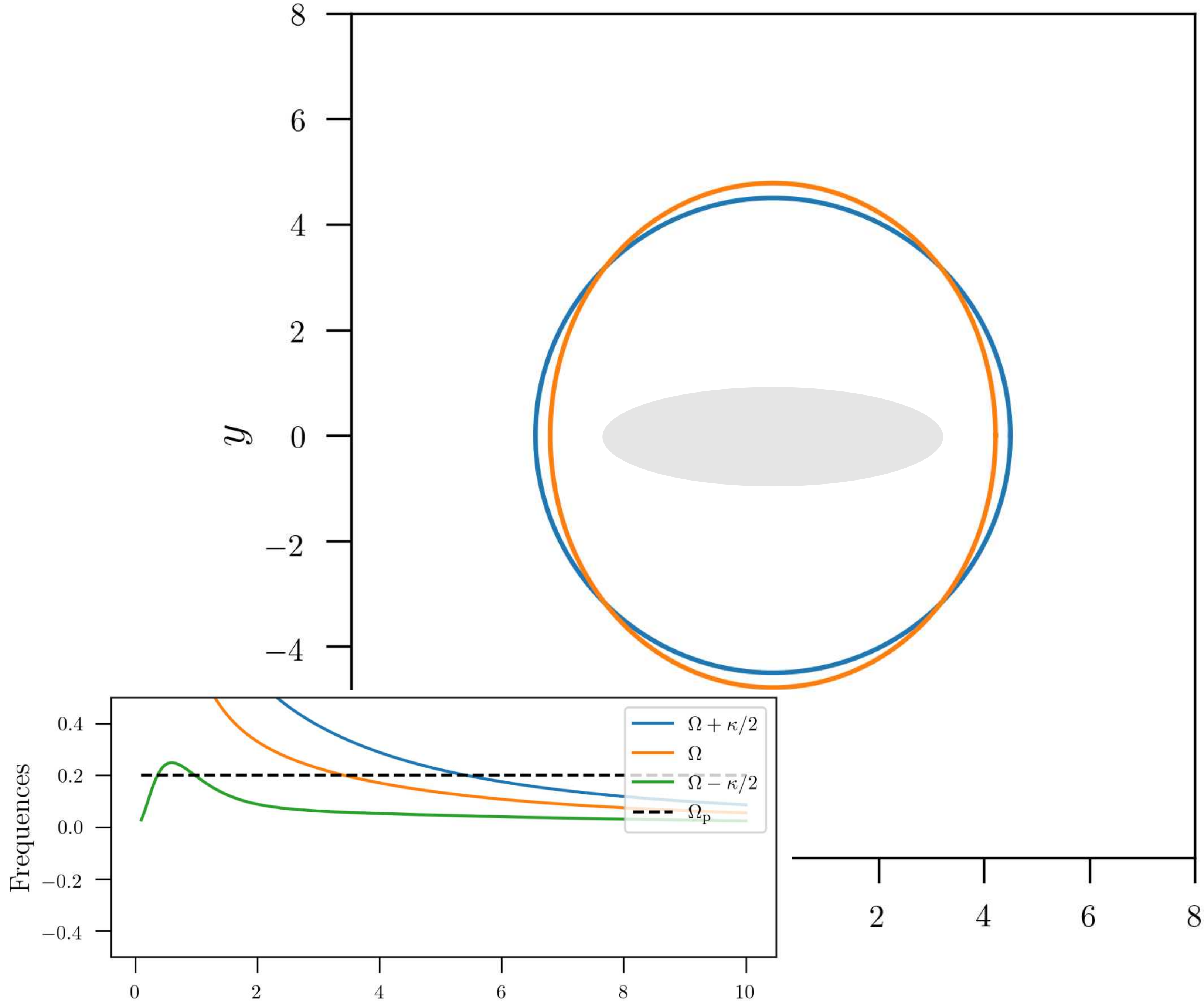


+ closed orbits
around L4, L5

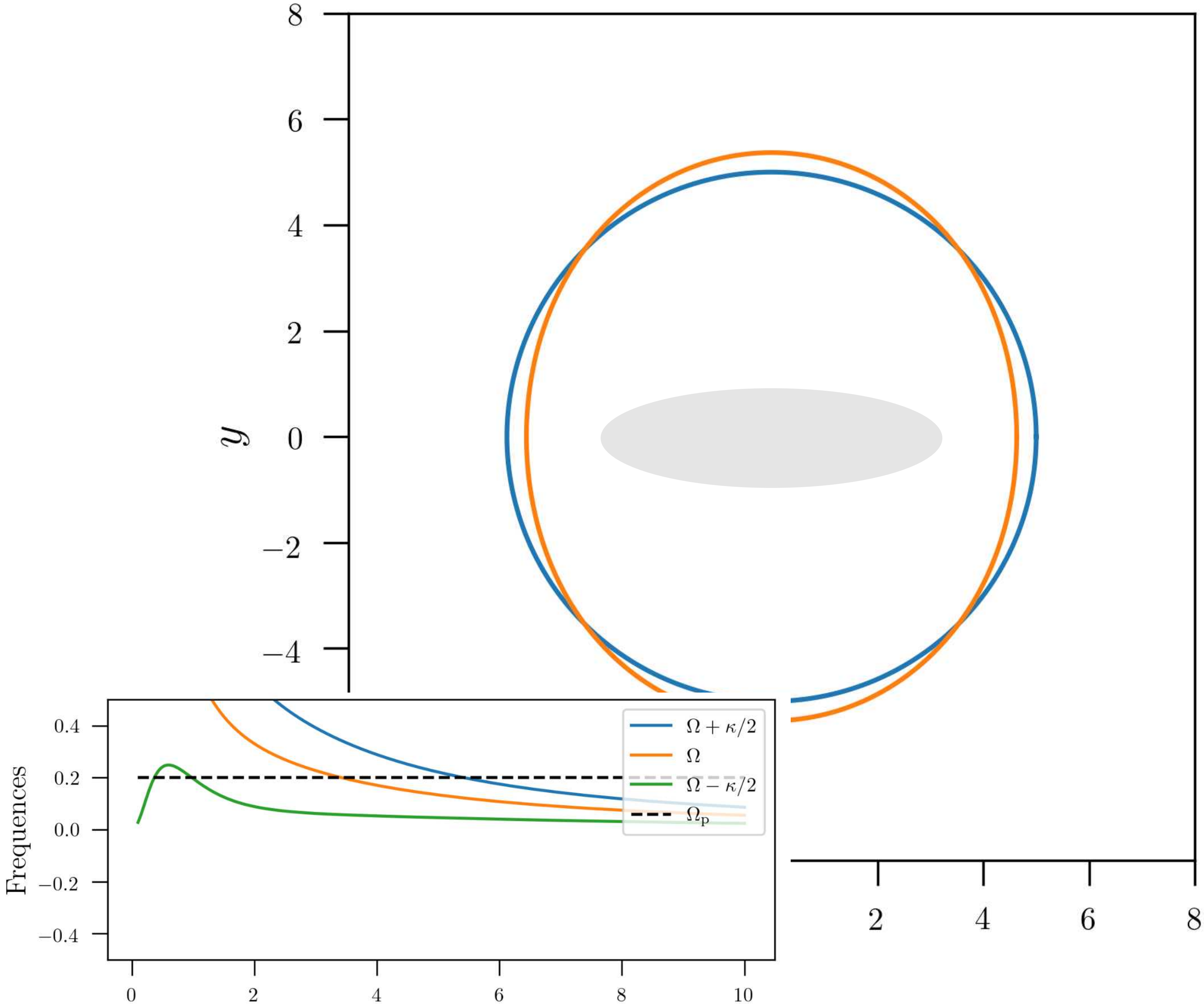
$R = 4.0$ $R_{\mathrm{CR}} < R < R_{\mathrm{OLR}}$



$R = 4.5$ $R_{\mathrm{CR}} < R < R_{\mathrm{OLR}}$

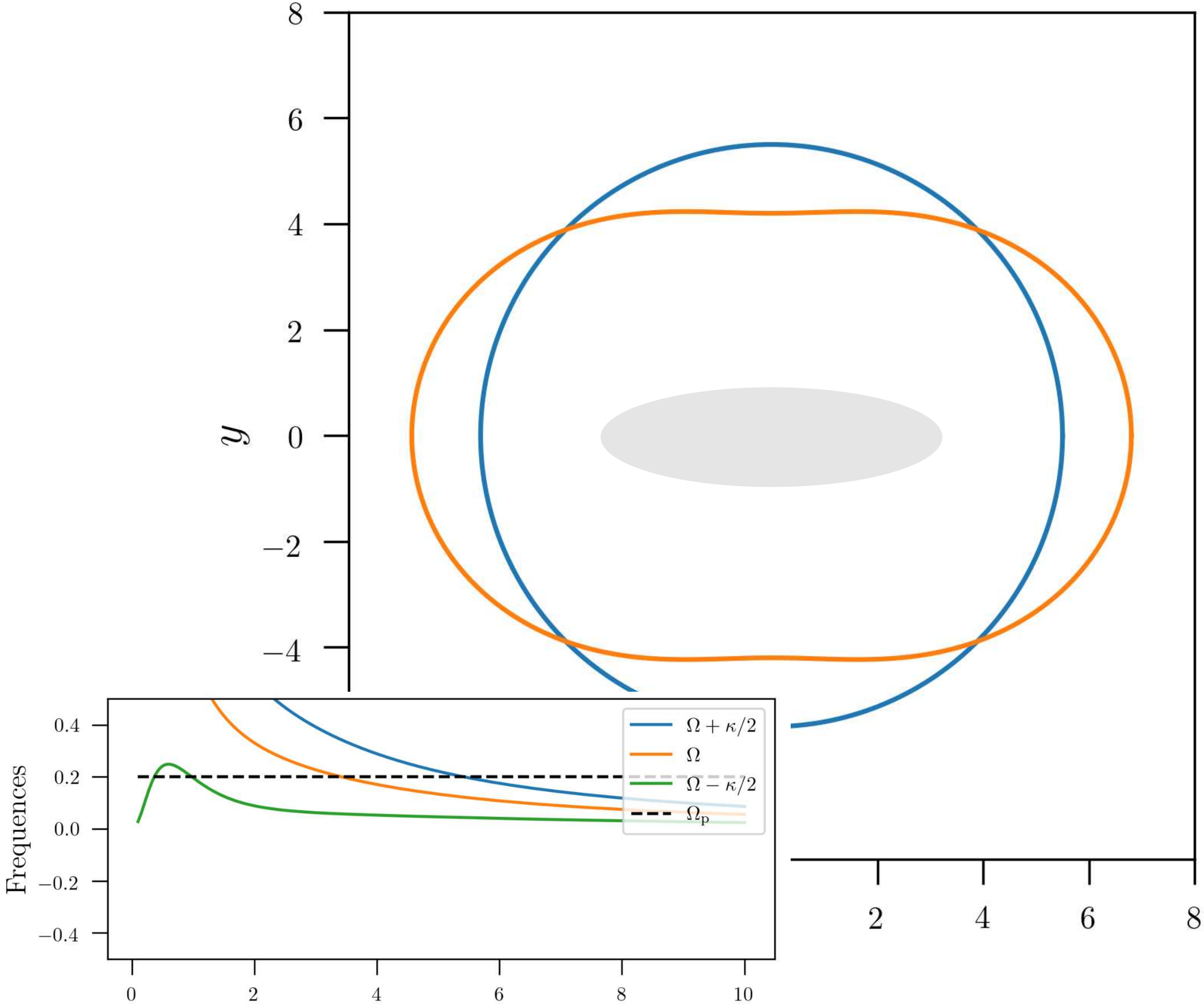


$R = 5.0$ $\quad R_{\mathrm{CR}} < R < R_{\mathrm{OLR}}$

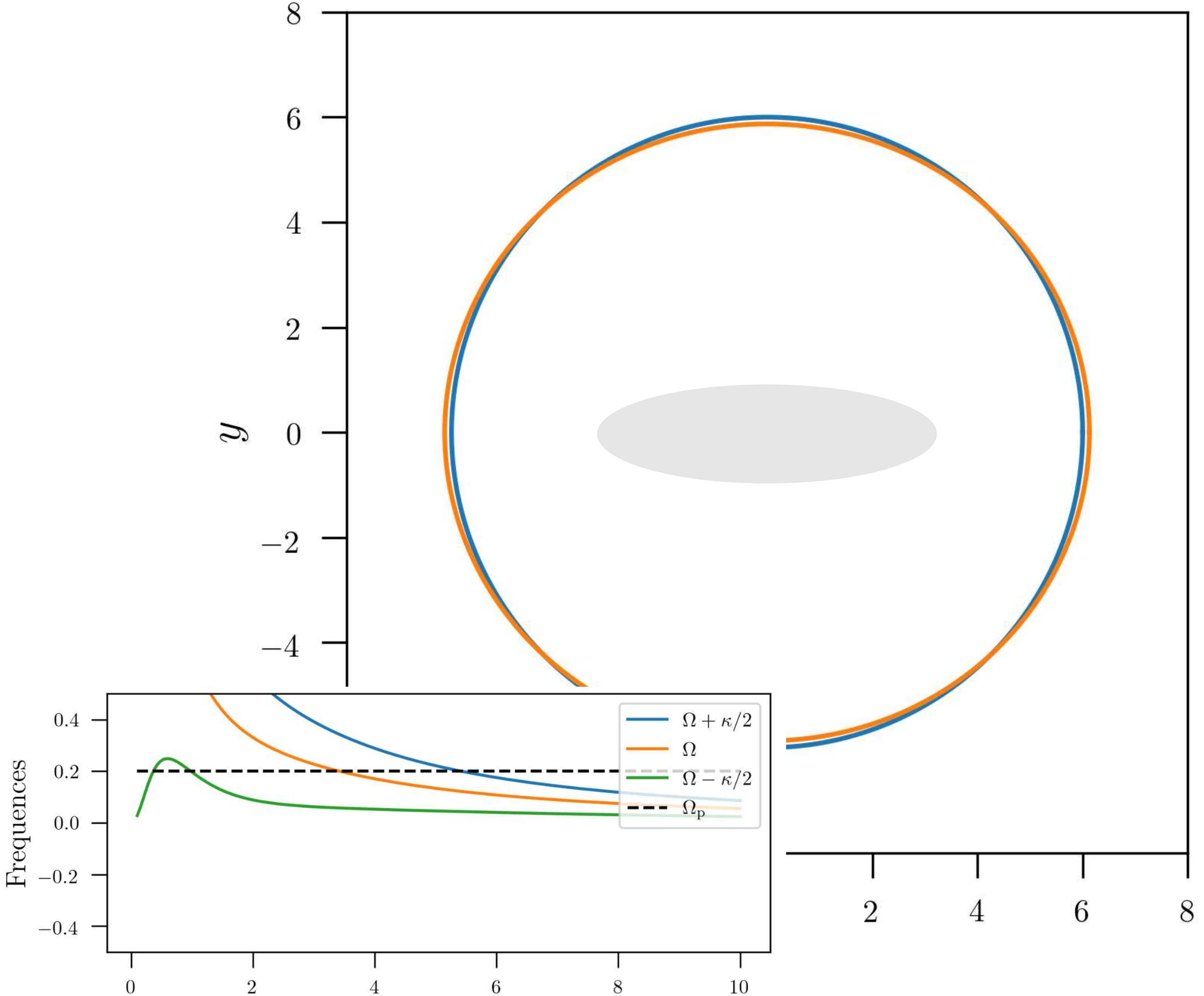


$R = 5.5$

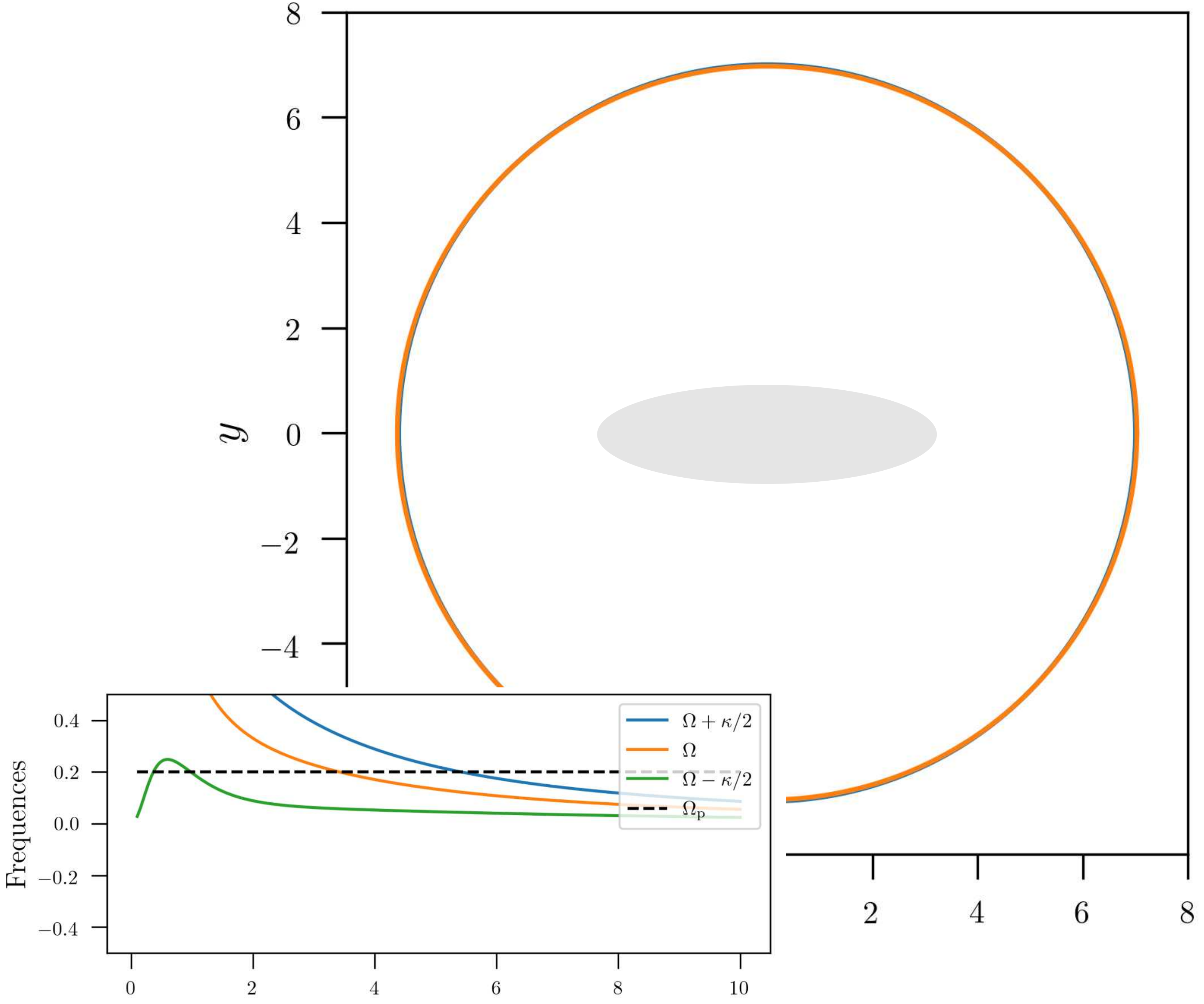
$R \cong R_{\mathrm{OLR}}$



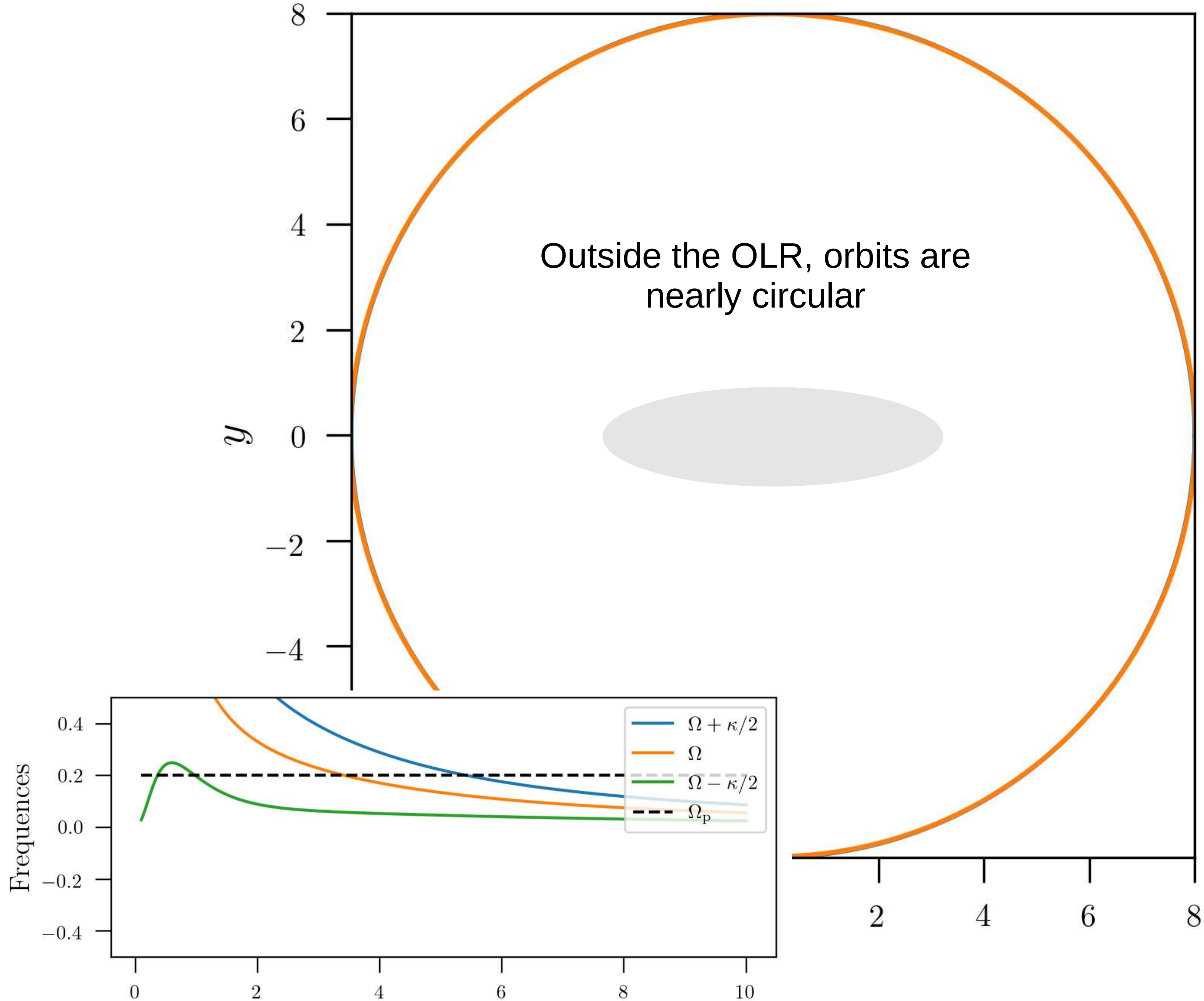
$R = 6.0$ $R_{\mathrm{OLR}} < R$



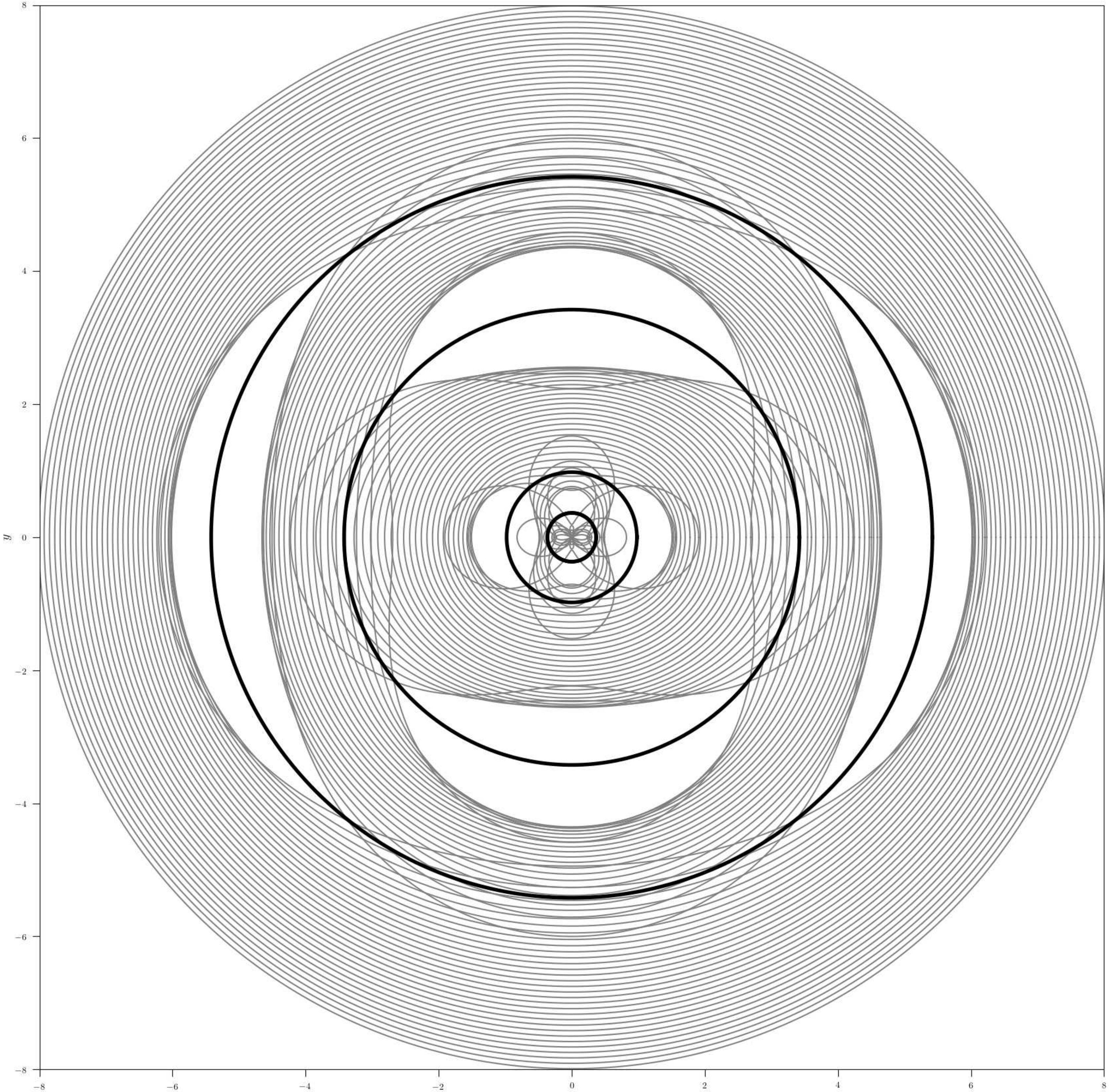
$R = 7.0$ $R_{\mathrm{OLR}} < R$



$R = 8.0$ $R_{\mathrm{OLR}} < R$

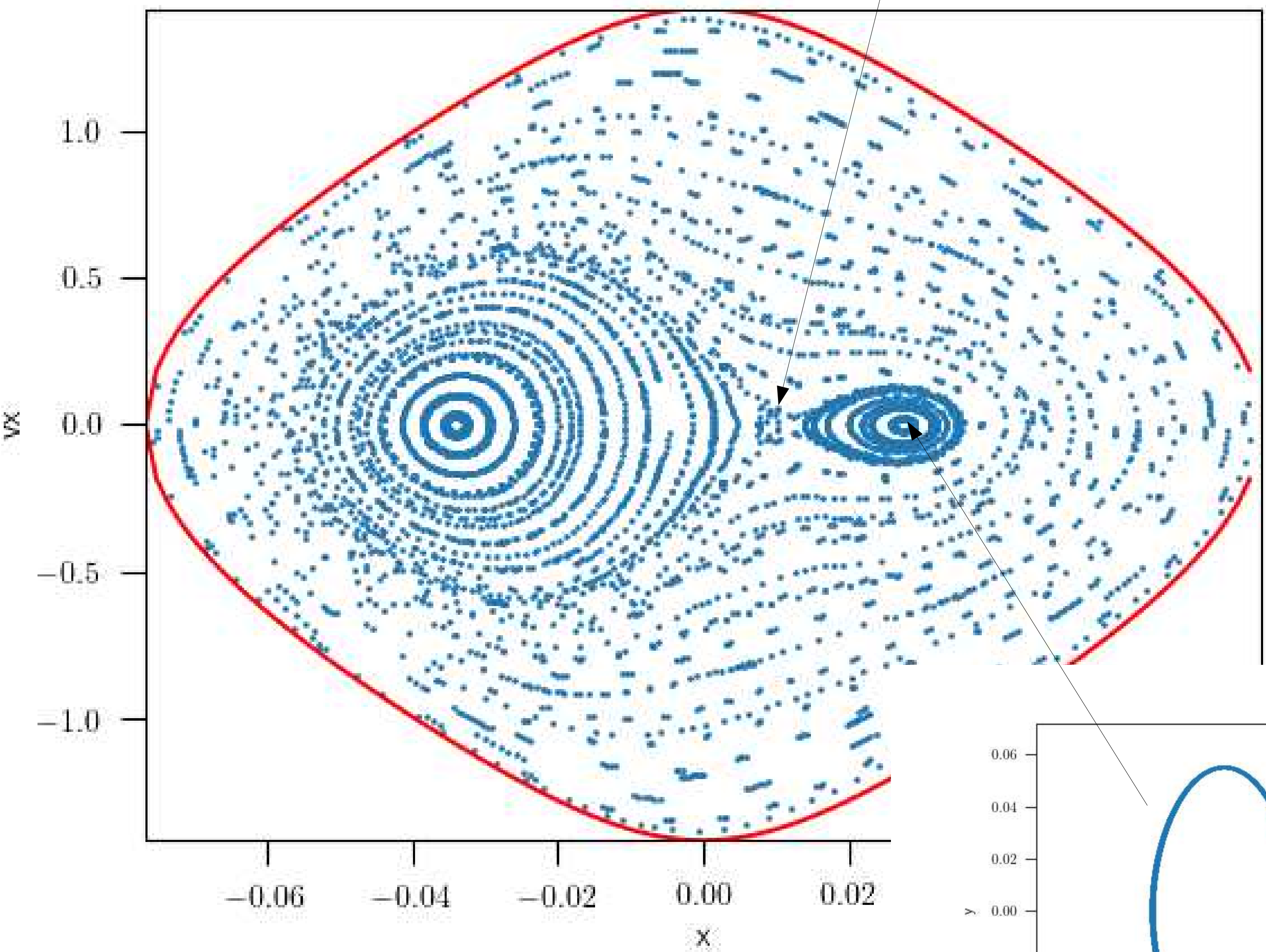


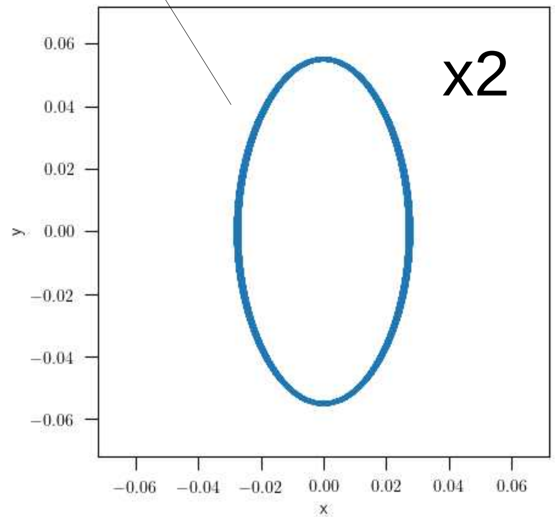
Outside the OLR, orbits are nearly circular



8
6
4
2
y 0
−2
−4
−6
−8
−8 −6 −4 −2 0 2 4 6 8

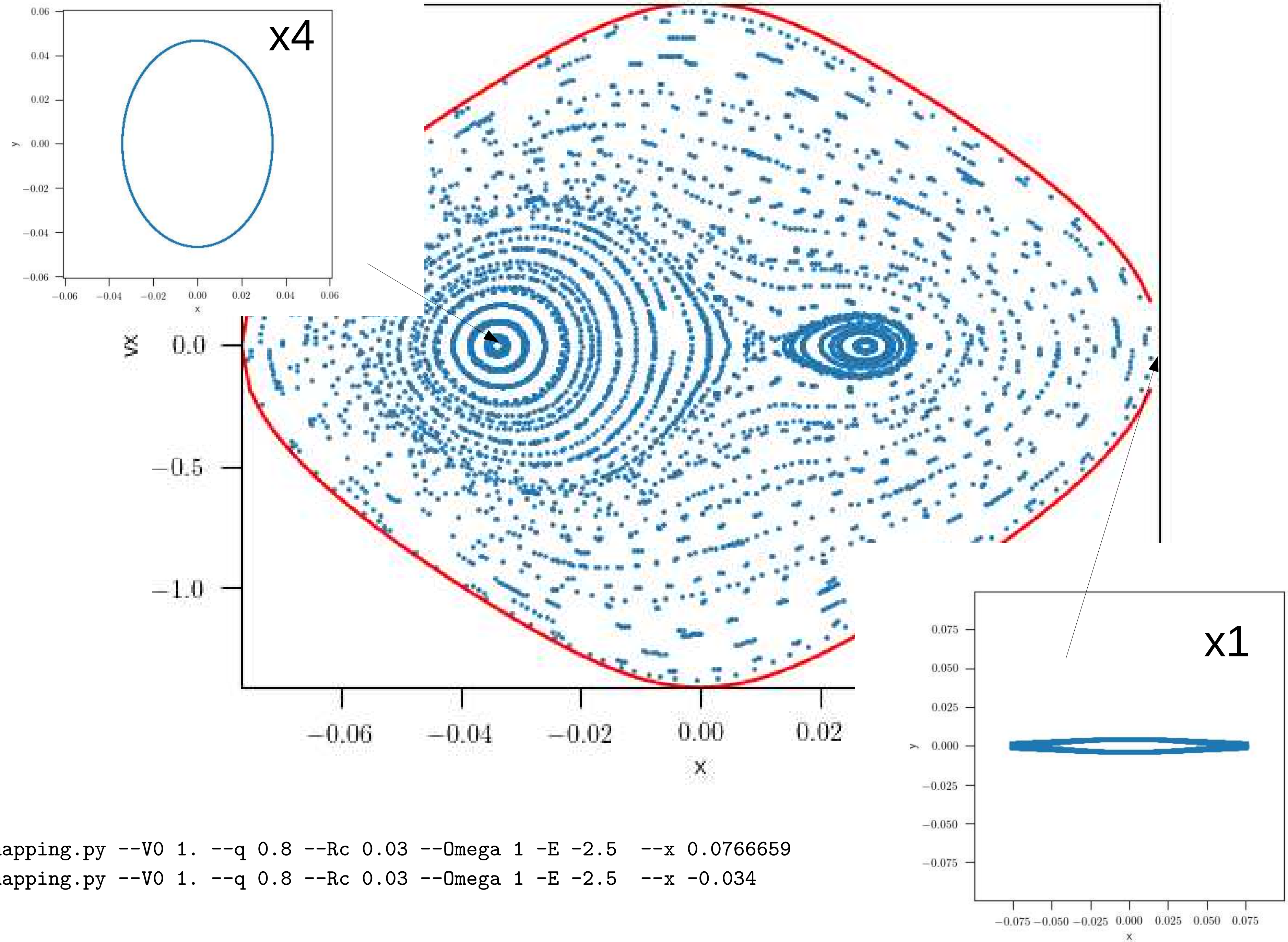
Bifurcation : apparition of x2 (stabe)/x3 (unstable) orbits





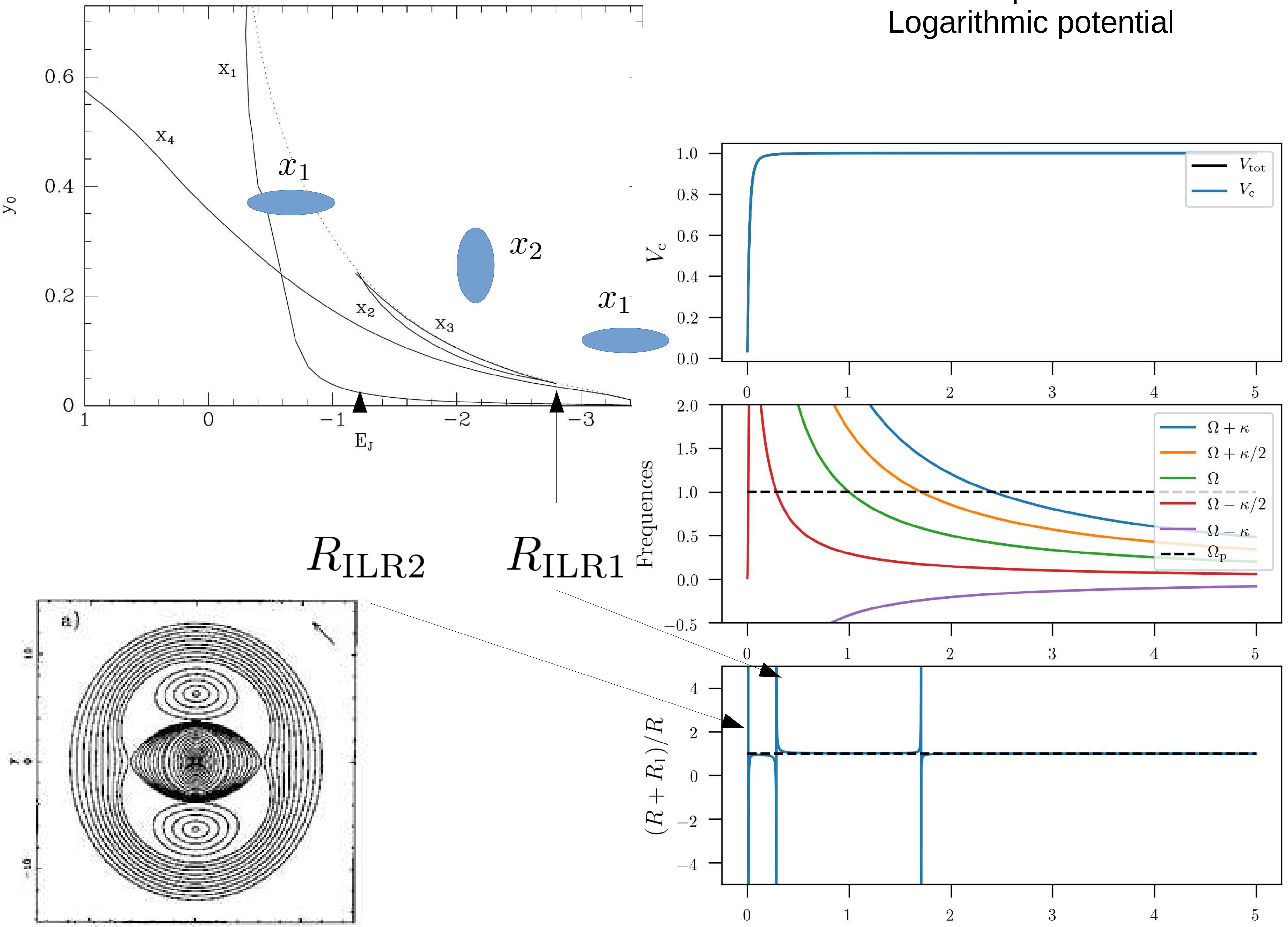

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0268
```

x1 : prograde x4 : retrograde




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x -0.034
```

Lindblad frequencies for the Logarithmic potential



Buta & Combes 1998

The collisionless Boltzmann equation

Introduction / Motivations

So far, we :

1. we modelled static potentials from a mass distribution (Poisson equation)

2. from the potential, we obtained forces and derived equations of motion leading us study orbits in different idealized potentials :

   - spherical potentials
   - axi-symmetric potentials (epicycles motions)
   - orbits in bared rotating potentials (motions around Lagrange points)

But :

1. We did not used any velocity constraints. We only used the positions of stars through the emission of light.

2. Nothing tells us that the models we used are at the equilibrium. This is not guarantee, if, for e.g., all velocities are zero...

3. We did do not include the self-gravity of the model or perturbations on it due to the orbits of stars.

Introduction / Motivations

Goal :

Build a self-consistent way galaxies, ensuring that they are at the equilibrium, i.e., if we compute the evolution of the galaxy under its own gravity, the evolution will be stationary.

→ requires the description of the density but also the velocity field

$$\rho(\vec{x}) \qquad \vec{v}(\vec{x})$$

Assumptions :

1. We will consider systems with a very large number of "particles" (stars, DM)

   → the collisionless approximation is valid
   → real orbits deviate not too much from the one predicted from the model (very large relaxation time)

We will seek for solution corresponding to $t_{\mathrm{relax}} = \infty$

2. We will consider systems composed of N identical particles, i.e., with all the same mass.

All particles will be equivalent

Introduction / Motivations

Goal :

Build a self-consistent way galaxies, ensuring that they are at the equilibrium, i.e., if we compute the evolution of the galaxy under its own gravity, the evolution will be stationary.

→ requires the description of the density but also the velocity field

$$\rho(\vec{x}) \qquad \vec{v}(\vec{x})$$

But :

It is impossible to describe analytically the orbits of billions of stars :

→ we need a probabilistic approach

Distribution function (DF)

Definition ① $f(\vec{x}, \vec{v}, t)$ or $f(\vec{w}, t)$ such that

$$f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3\vec{v} \quad \text{or} \quad f(\vec{w}, t)\, d^3\vec{w}$$

is the probability that at the time t, a randomly chosen star "i" has its position $\vec{x}_i$ an velocity $\vec{v}_i$, or phase space coordinates $\vec{w}_i$ in the ranges

$$\vec{x}_i \in [\vec{x}, \vec{x} + d\vec{x}]$$
$$\vec{v}_i \in [\vec{v}, \vec{v} + d\vec{v}]$$
$$\equiv \quad \vec{w}_i \in [\vec{w}, \vec{w} + d\vec{w}]$$

obviously: (normalisation)

$$\int f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3 v = 1$$
$$\equiv \int f(\vec{w}, t)\, d^6\vec{w} = 1$$

the particle is for sure somewhere in the phase space

$f(\vec{x}, \vec{v}, t)$ is the probability density of the phase space.

Distribution function (DF)

Definition ② $\tilde{f}(\vec{x}, \vec{v}, t)$ or $\tilde{f}(\vec{w}, t)$ such that

$\tilde{f}(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3\vec{v}$ or $\tilde{f}(\vec{w}, t)\, d^3\vec{w}$

is the number of stars having position $\vec{x}$ and velocities $\vec{v}$ ($\vec{w}$) in the intervals at time t:

$$\vec{x}_i \in [\vec{x}, \vec{x} + d\vec{x}]$$
$$\vec{v}_i \in [\vec{v}, \vec{v} + d\vec{v}]$$
$$\vec{w}_i \in [\vec{w}, \vec{w} + d\vec{w}]$$

obviously: (normalisation)

$$\int \tilde{f}(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3 v = N$$

$$\equiv \int \tilde{f}(\vec{w}, t)\, d^6\vec{w} = N$$

The are exactly N particles in the phase space

$\tilde{f}(\vec{x}, \vec{v}, t)$ is the number density of the phase space.

Combining Def. ① and Def ②

$$N f(\vec{x}, \vec{v}, t) = \hat{f}(\vec{x}, \vec{v}, t)$$

Notes

- we will sometimes forget the "~"
- the time dependence "t" will not be sytematically written

Using definition ①

The probability of finding a star "i" in the subvolume of the phase space ν is :

$$P = \int_{\nu} f(\vec{w})\, d^6\vec{w}$$

However, imagine that we are using another canonical coordinate system $\vec{W}$ (in which the Hamilton equations are valid)

e.g. $(x, y, p_x = \dot{x}, p_y = \dot{y}) \to (r, \theta, p_r = \dot{r}, p_\theta = r^2\dot{\theta})$

$$P^W = \int_{\nu} F(\vec{W})\, d^6\vec{W} = P$$

The probability must not be affected by a coordinate change.

If ν is taken small enough, we can assume $\rho(\bar{w})$ and $F(\hat{W})$ to be constant and hence

$$\rho(\bar{w}_\nu) \int_\nu d^6\bar{w} = F(\hat{W}_\nu) \int_\nu d^6\bar{W}$$

But, for canonical coordinates, the phase space volume element is the same:

$$\int_\nu d^6\bar{w} = \int_\nu d^6\bar{W}$$

Thus

$$\rho(\bar{w}) = F(\bar{W})$$

The density of the phase space is independent of the coordinate system

Corollary: We can use any system of canonical coordinates $\bar{w} = (\bar{q}, \bar{p})$ to define the distribution function

The collisionless Boltzmann equation

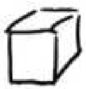
- What is the evolution of $f(\vec{w})$ over time ?

As the mass, the probability is a conserved quantity. $\rho = N f$

the number of stars is a conserved quantity.

in the phase space

Continuity equation (similar than for hydrodynamics)

V  the time variation of the mass in V : $\frac{dM}{dt} = \sum_{faces} \rho \vec{V} \cdot d\vec{S}$

mass flux

Gauss

Mass conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_x \cdot (\rho \vec{v}) = 0$$

mass flux through the surface of the volume

Probability conservation

$$\frac{\partial f}{\partial t} + \vec{\nabla}_w \cdot (f \dot{\vec{w}}) = 0$$

probability flux through the surface of the volume

Analogy with the continuity equation in hydrodynamics

$\rho(\vec{x}, t)$ $\qquad \vec{v} = \frac{d}{dt}\vec{x}$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_{\vec{x}}(\rho \vec{v}) = 0$$

Lagrangian derivative

$$\frac{d}{dt}\rho(\vec{x}, t) = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}_x \rho$$

Flux divergence

$$\vec{\nabla}_{\vec{x}}(\rho \vec{v}) = \vec{v} \cdot \vec{\nabla}_{\vec{x}} \rho + \rho \vec{\nabla}_{\vec{x}} \cdot \vec{v}$$

$$\vec{v} \cdot \vec{\nabla}_x \rho = \vec{\nabla}_{\vec{x}}(\rho \vec{v}) - \rho \vec{\nabla}_{\vec{x}} \cdot \vec{v}$$

$\rho(\hat{\omega}, t)$ $\qquad \dot{\hat{\omega}} = \frac{d}{dt}\hat{\omega}$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_{\omega}(\rho \dot{\hat{\omega}}) = 0$$

Lagrangian derivative

$$\frac{d}{dt}\rho(\hat{\omega}, t) = \frac{\partial \rho}{\partial t} + \dot{\hat{\omega}} \cdot \vec{\nabla}_{\omega} \rho$$

Flux divergence

$$\vec{\nabla}_{\hat{\omega}}(\rho \dot{\hat{\omega}}) = \dot{\hat{\omega}} \cdot \vec{\nabla}_x \rho + \rho \vec{\nabla}_{\omega} \cdot \dot{\hat{\omega}}$$

$$\dot{\hat{\omega}} \cdot \vec{\nabla}_x \rho = \vec{\nabla}_{\hat{\omega}}(\rho \dot{\hat{\omega}}) - \rho \vec{\nabla}_{\omega} \cdot \dot{\hat{\omega}}$$

Lagrangian derivative

$$\frac{d}{dt}\rho(\vec{x},t) = \frac{\partial \rho}{\partial t} + \vec{v}\cdot\vec{\nabla}_x \rho$$

$$= \underbrace{\frac{\partial \rho}{\partial t} + \vec{\nabla}_x(\rho\vec{v})}_{=0} - \rho\vec{\nabla}_x\vec{v}$$

= 0 continuity Eqn.

$$\boxed{\frac{d}{dt}\rho(\vec{x},t) = -\rho\vec{\nabla}_x\vec{v}}$$

the increase of ρ along the flow is due to compression

incompressible fluid :

$$\vec{\nabla}_x\vec{v} = 0$$

Lagrangian derivative

$$\frac{d}{dt}\rho(\vec{w},t) = \frac{\partial \rho}{\partial t} + \dot{\vec{w}}\cdot\vec{\nabla}_w \rho$$

$$= \underbrace{\frac{\partial \rho}{\partial t} + \vec{\nabla}_w(\rho\dot{\vec{w}})}_{=0} - \underbrace{\rho\vec{\nabla}_w\dot{\vec{w}}}_{=0}$$

= 0 continuity Eqn.

= 0 canonical coords. (replace $\dot{\vec{w}}$ with Hamilton equations)

$$\boxed{\frac{d}{dt}\rho(\vec{w},t) = 0}$$

⟹ behaves like an incompressible fluid

The flow through the phase space is incompressible

Seen from an observer that follow the flow in the phase space i.e. an orbit : ρ is constant

(See Ogorodnikov)

Liouville's theorem (corollary)

In the motion of a stellar system, any volume of phase space remains constant

$d\nu$: an infinitesimal volume of the phase space

$dN(t)$: the numer of stars in $d\nu(t)$ at t

$$dN(t) = \tilde{f}(\tilde{w}, t)\, d\nu(t)$$

$dN(t')$: the numer of stars in $d\nu(t')$ at t'

$$dN(t') = \tilde{f}(\tilde{w}, t')\, d\nu(t')$$

But $dN(t) = dN(t')$

Because EoM are 1st order differential equations, only the points that were in $d\nu$ at t are in $d\nu'$ at t'

$$\equiv \quad \frac{dN}{dt} = 0$$

Thus

$$\frac{dN}{dt} = \frac{d}{dt}\left(\tilde{\rho}(\omega, t)\, d\nu(t) \right)$$

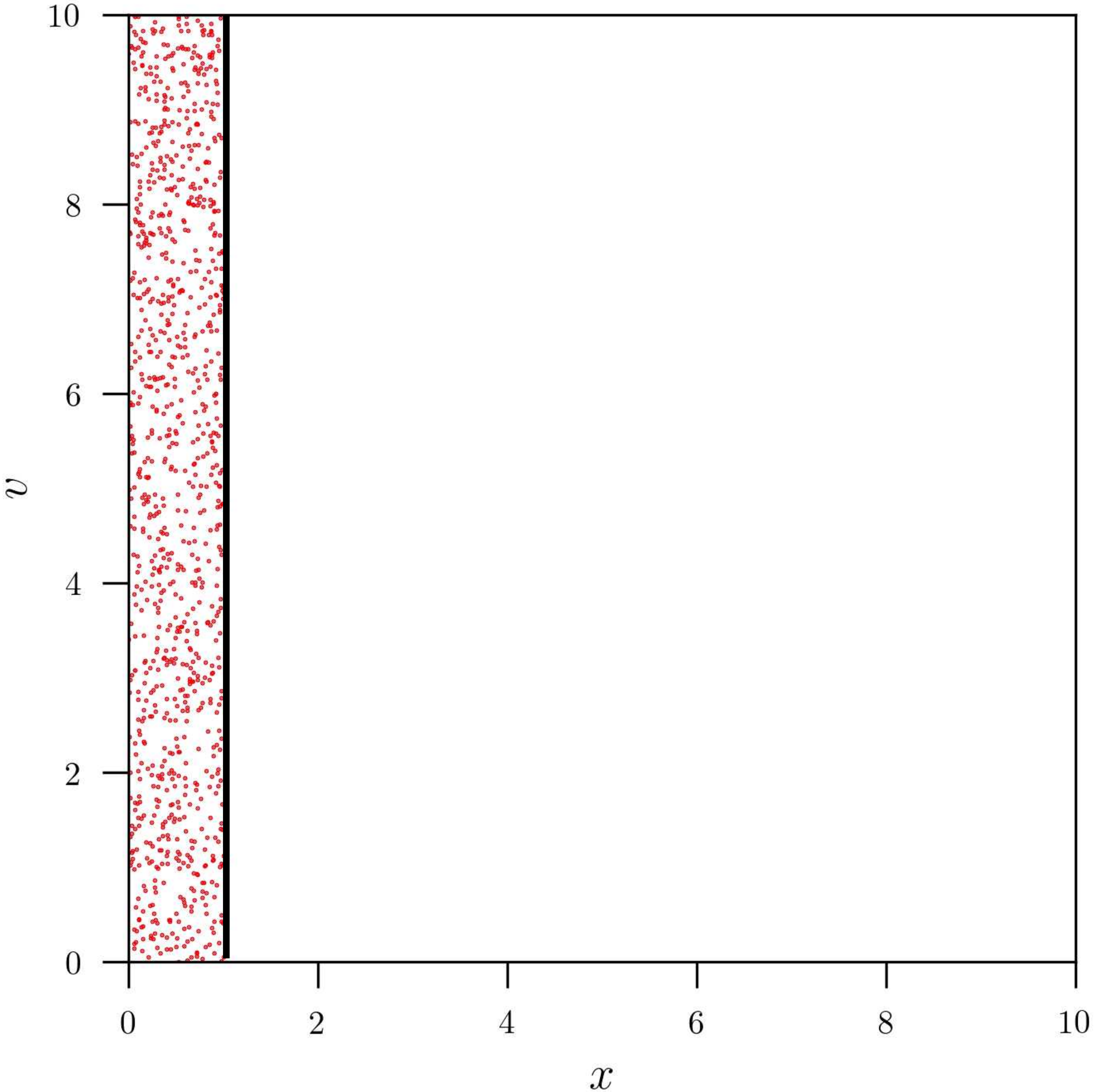
$$= \underbrace{\frac{d}{dt}\left(\tilde{\rho}(\omega, t) \right)}_{=0 \text{ (Boltzmann equation)}} d\nu(t) + \tilde{\rho}(\omega, t)\, \frac{d}{dt}\left(d\nu(t) \right) = 0$$

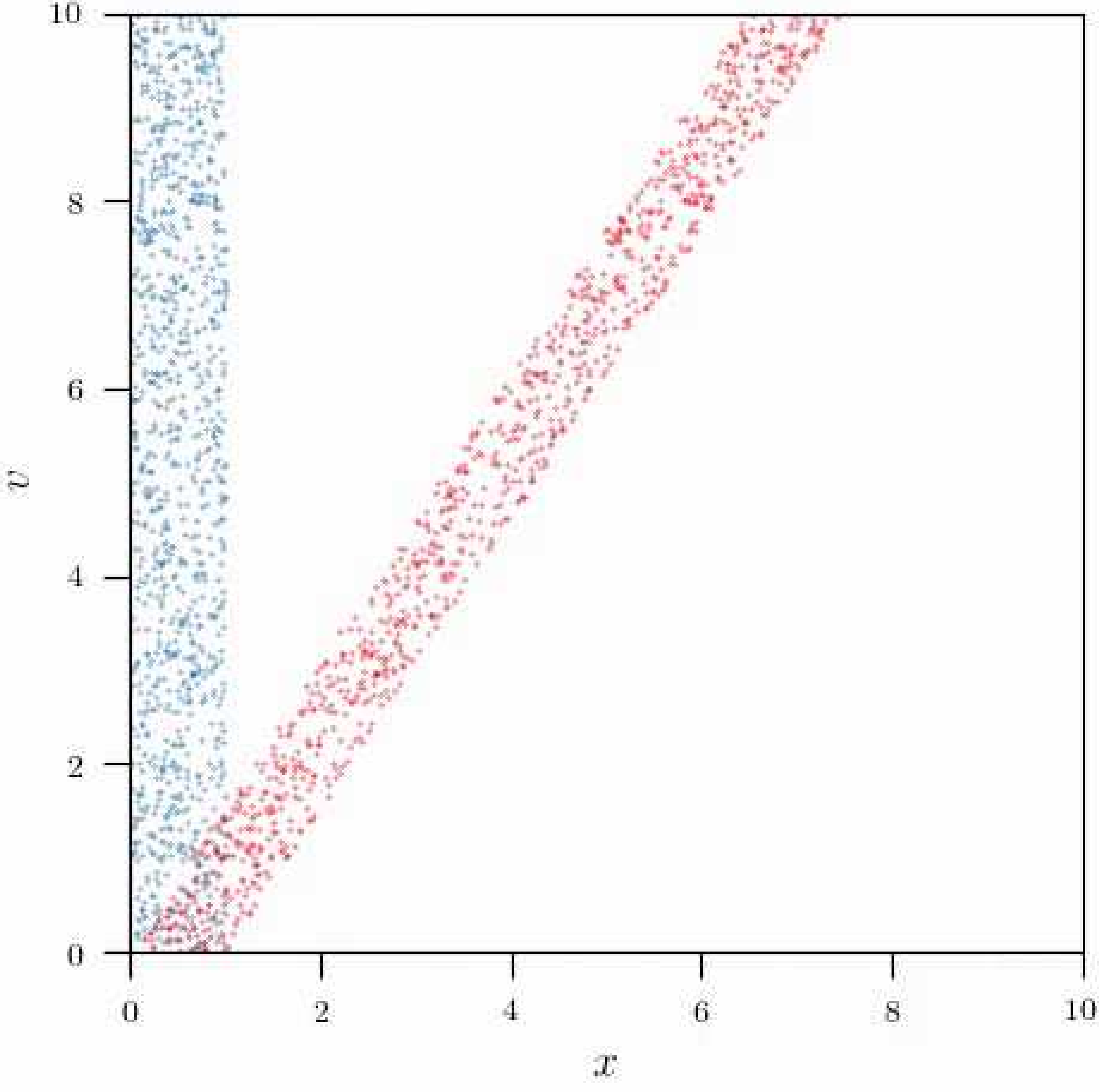
$$\Rightarrow \quad \frac{d}{dt}\left(d\nu(t) \right)$$

$$\boxed{dV(t) = \text{cte}}$$

The distribution function remains constant along the flow

Illustration 1 : Ideal race: each runner has a constant speed

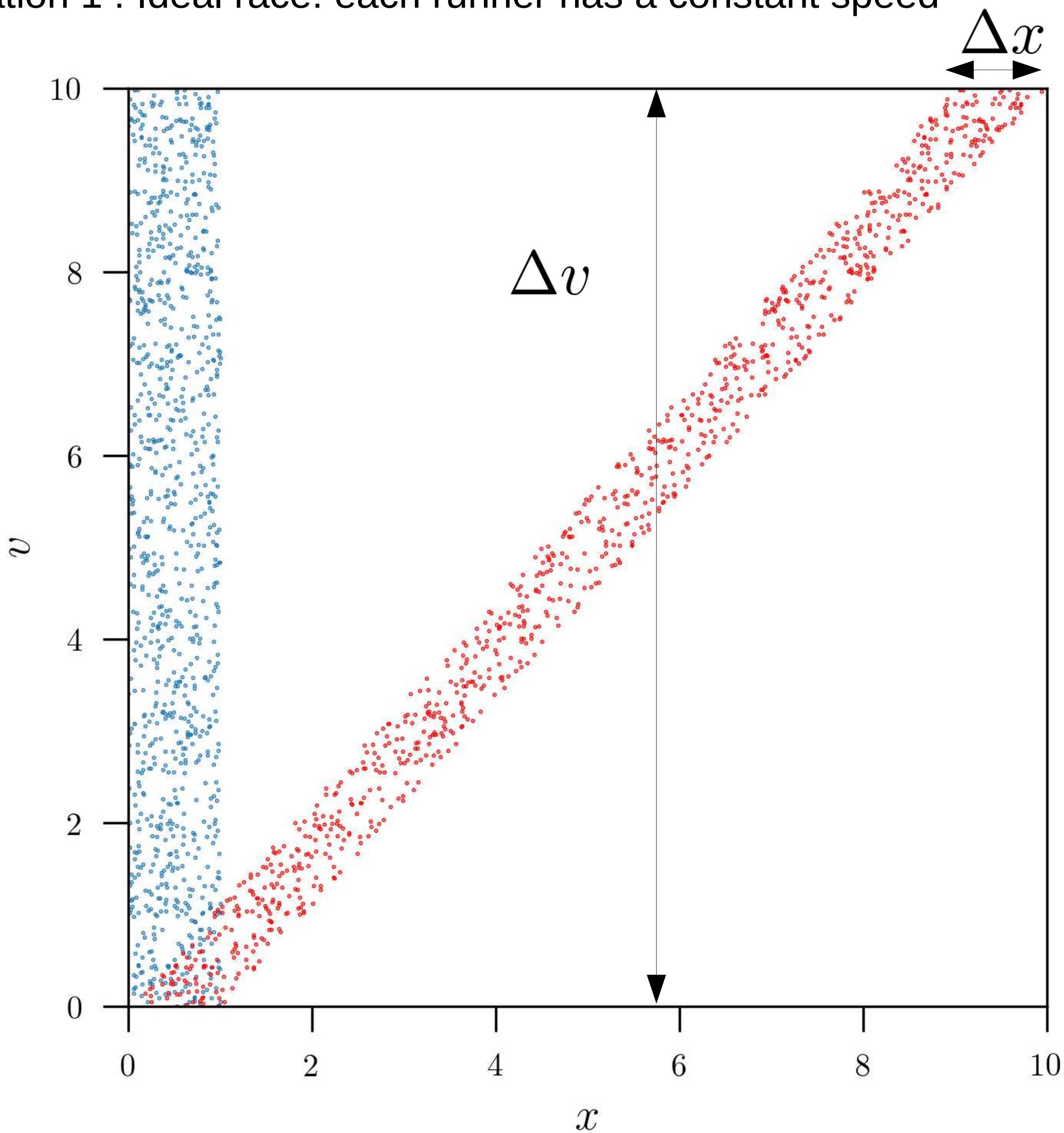




10
8
6
4
2
0
v
0
2
4
6
8
10
x

The distribution function remains constant along the flow

Illustration 1 : Ideal race: each runner has a constant speed

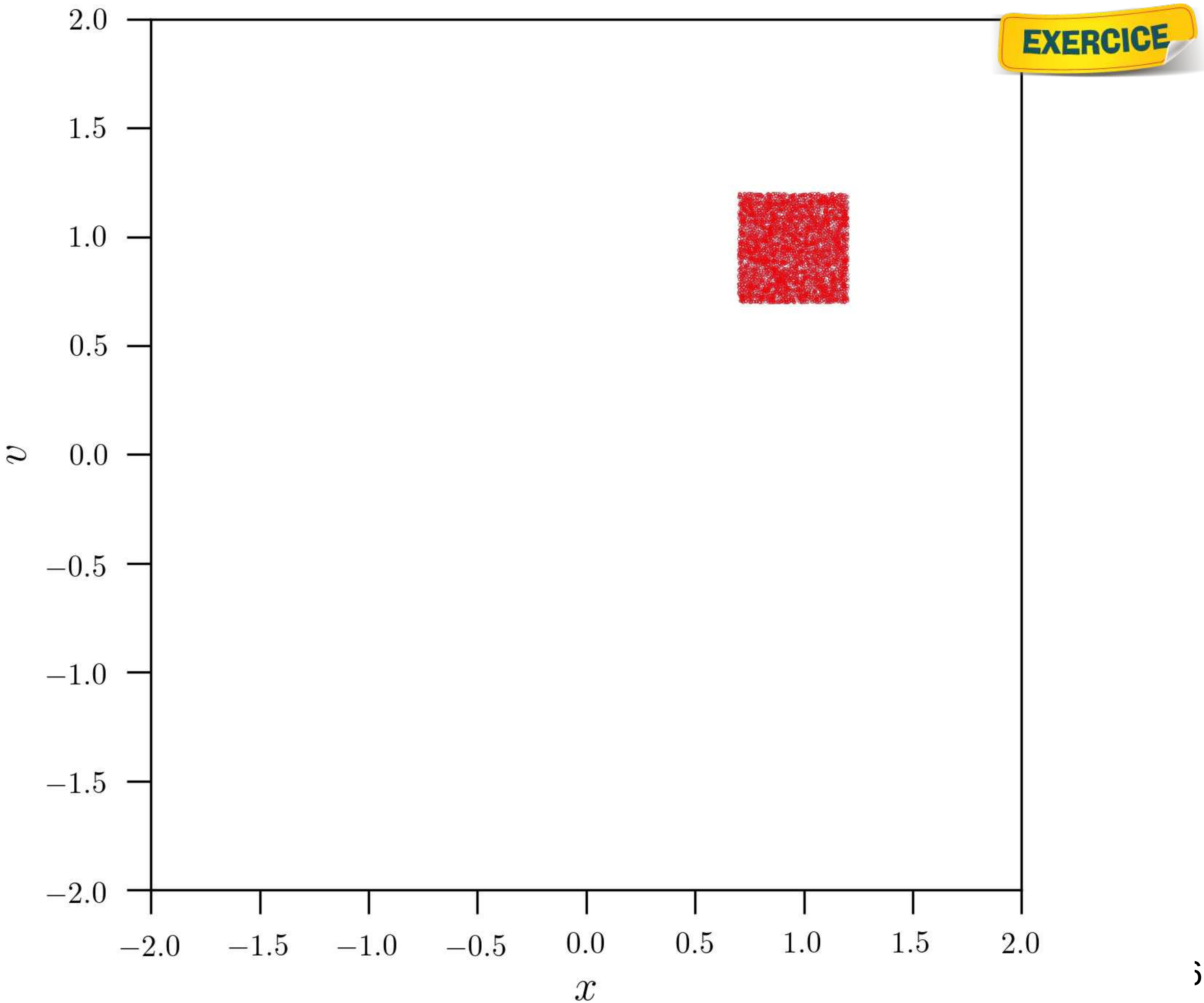


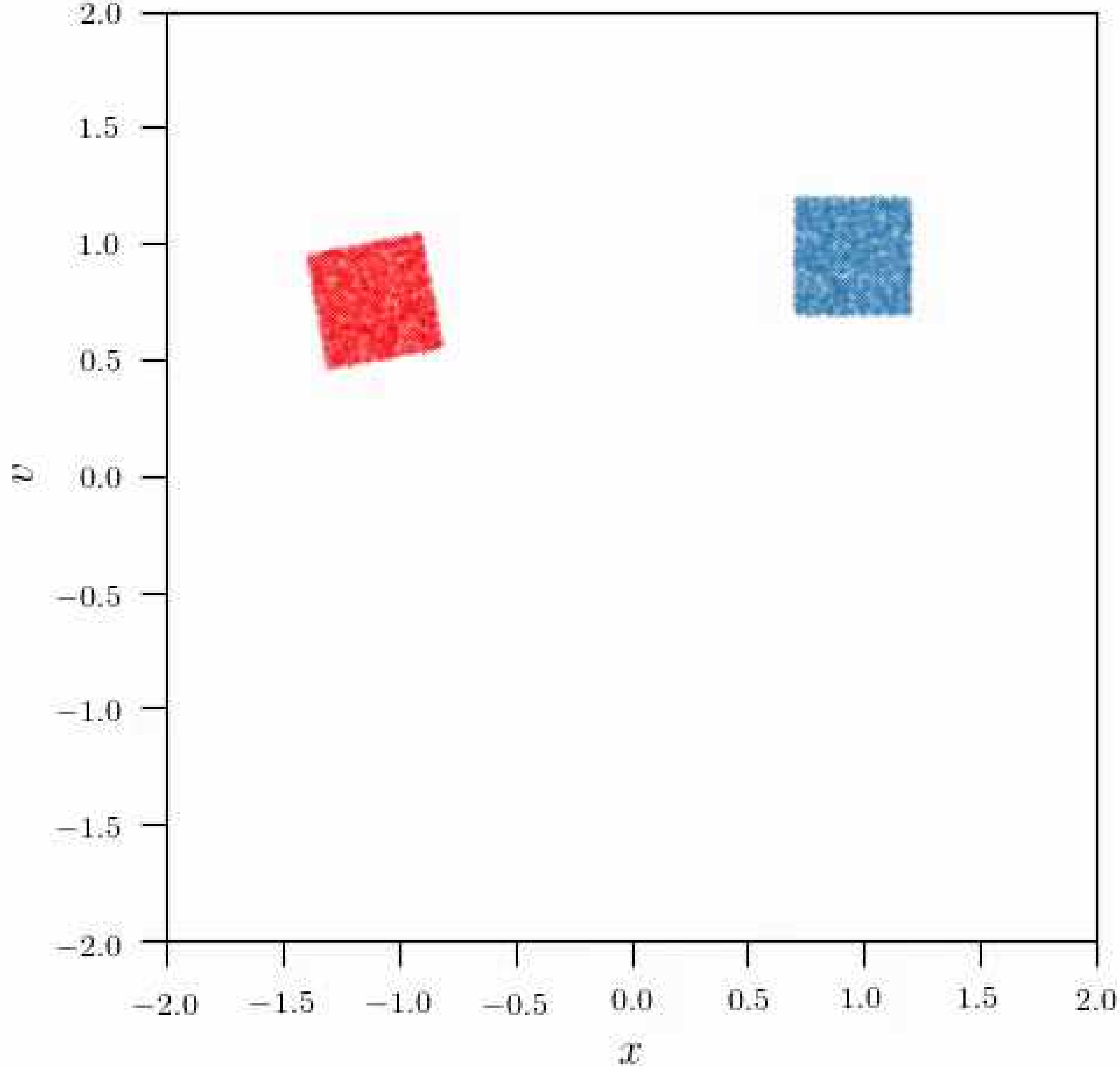
ν: the phase space volume

$$\tilde{f}(t=0) = \frac{N}{\nu_0} = \frac{N}{\Delta x \Delta v}$$

$$\tilde{f}(t=t) = \frac{N}{\nu_t} = \frac{N}{\Delta x \Delta v}$$

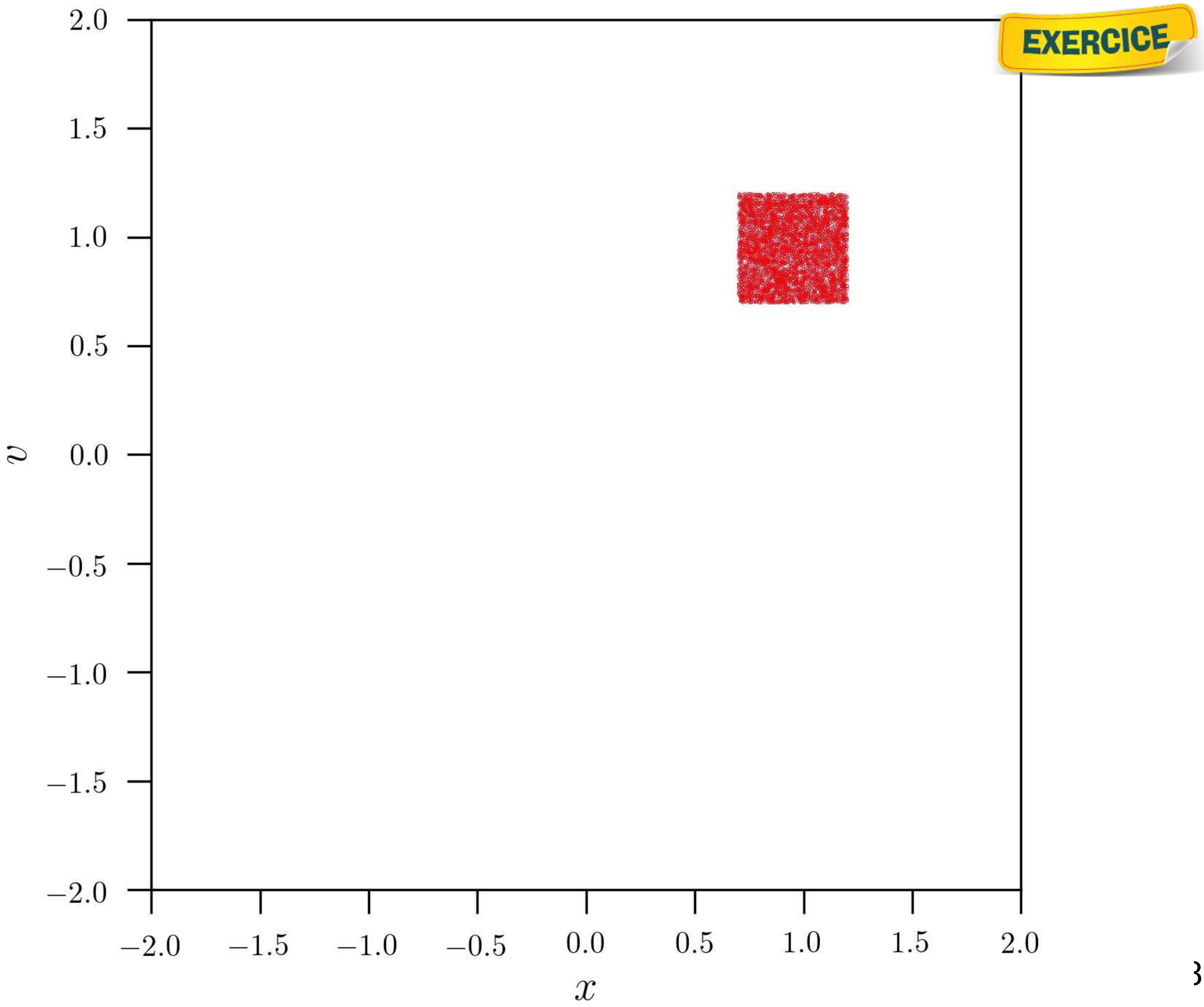
Illustration 2 : Harmonic oscillator $H(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$ $\omega = 1$





2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
v
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x

Illustration 2 : Harmonic oscillator $H(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$ $\omega = 0.75$



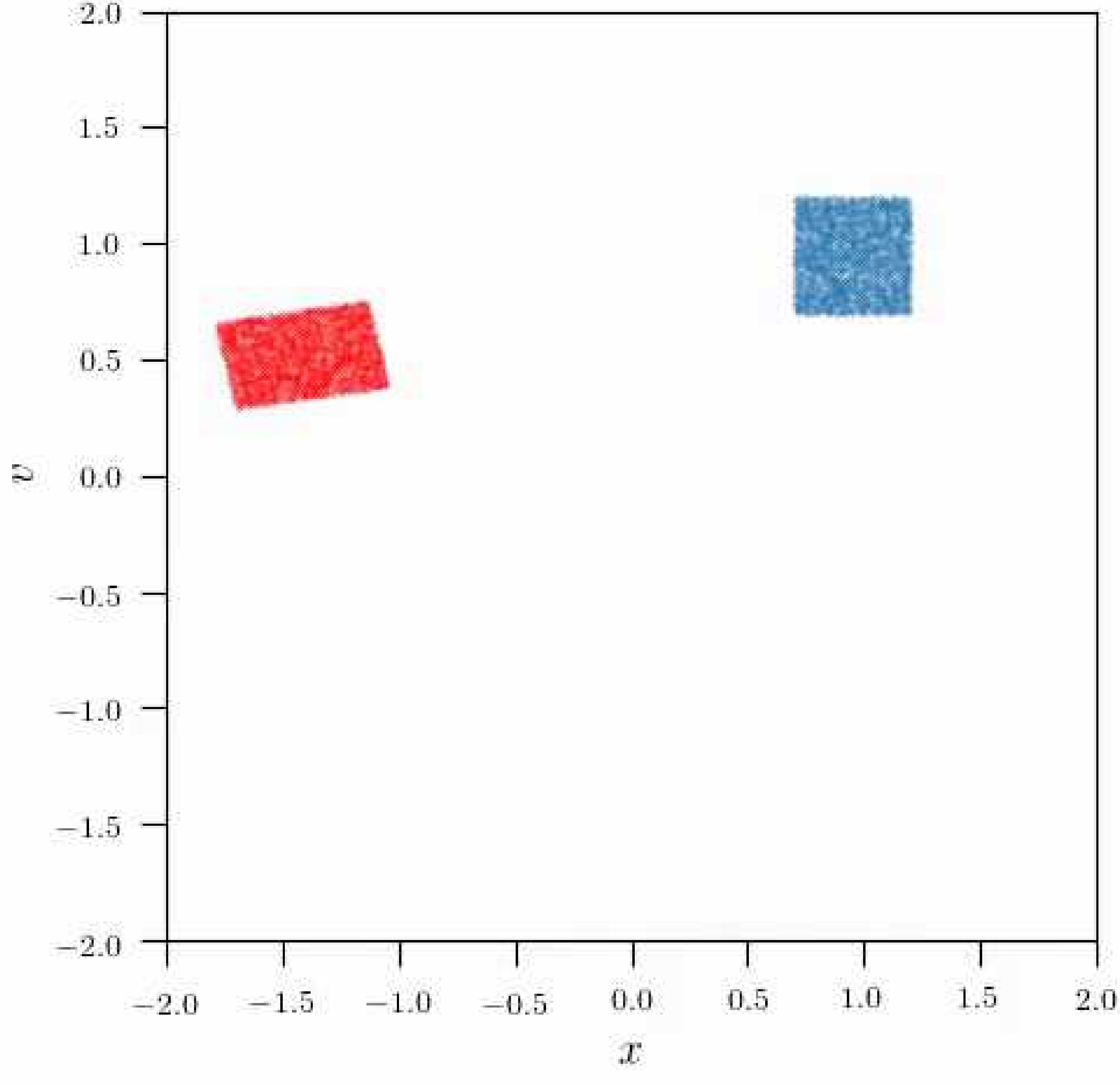
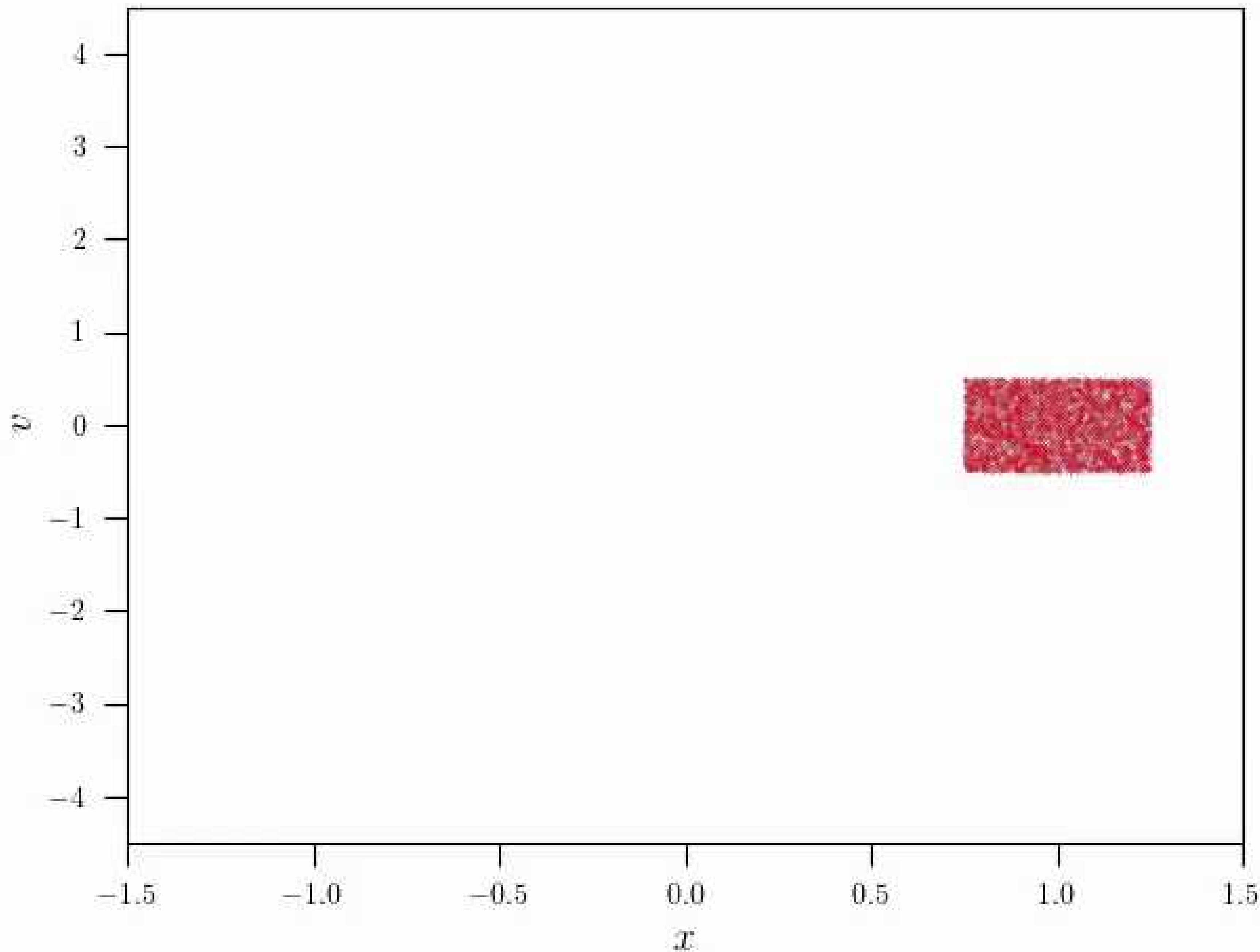

2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
y
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x

Illustration 3 : Plummer



Expressing the continuity equation using $\vec{\omega} = (\vec{q}, \vec{p})$

$$\frac{d}{dt}\rho(\vec{\omega},t) = \frac{\partial \rho(\vec{\omega},t)}{\partial t} + \vec{\nabla}_{\vec{\omega}}\left(\rho(\vec{\omega},t)\,\dot{\vec{\omega}}\right) = 0$$

$$= \frac{\partial \rho(\vec{\omega},t)}{\partial t} + \dot{\vec{\omega}}\,\vec{\nabla}_{\vec{\omega}}\left(\rho(\vec{\omega},t)\right) = 0$$

$$= \frac{\partial \rho(\vec{q},\vec{p})}{\partial t} + \sum_i \dot{q}_i \frac{\partial}{\partial q_i}\rho(\vec{q},\vec{p}) + \sum_i \dot{p}_i \frac{\partial}{\partial p_i}\rho(\vec{q},\vec{p})$$

$$\frac{d}{dt}\rho(\vec{\omega},t) = \frac{\partial \rho(\vec{q},\vec{p})}{\partial t} + \dot{\vec{q}}\,\frac{\partial}{\partial \vec{q}}\rho(\vec{q},\vec{p}) + \dot{\vec{p}}\,\frac{\partial}{\partial \vec{p}}\rho(\vec{q},\vec{p}) = 0$$

The Collisionless Boltzmann Equation

Using the Hamilton Equations

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}} \qquad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}$$

Then
$$\frac{\partial}{\partial t} f + \dot{\vec{q}} \frac{\partial}{\partial \vec{q}} f + \dot{\vec{p}} \frac{\partial}{\partial \vec{p}} f = 0$$

becomes
$$\frac{\partial}{\partial t} f + \frac{\partial H}{\partial \vec{p}} \frac{\partial f}{\partial \vec{q}} - \frac{\partial H}{\partial \vec{q}} \frac{\partial f}{\partial \vec{p}} = 0$$

$$\frac{\partial}{\partial t} f + [f, H] = 0$$

Poisson brackets
$$[A, B] := \frac{\partial A}{\partial \vec{q}} \frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}} \frac{\partial B}{\partial \vec{q}}$$
$$= \sum_i^n \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i}$$

The Collisionless Boltzmann equation in various coordinates

EXERCICE

Generalized coordinates

$$\vec{p} = \frac{\partial L(\vec{q}, \vec{p})}{\partial \dot{\vec{q}}}$$

$$\frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \frac{\partial f}{\partial \vec{q}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{q}} \cdot \frac{\partial H}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{q}} = 0$$

Cartesian coordinates

$$\begin{cases} p_x = \dot{x} = v_x \\ p_y = \dot{y} = v_y \\ p_z = \dot{z} = v_z \end{cases} \qquad H = \frac{1}{2}\left(v_x^2 + v_y^2 + v_z^2\right) + \Phi(x, y, z)$$

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

Spherical coordinates

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2 \dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta \dot{\phi}) = r \sin(\theta)\, v_\phi \end{cases} \qquad H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2(\theta)}\right) + \Phi(R, \theta, \phi)$$

$$\frac{\partial f}{\partial t} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{p_\phi}{r^2 \sin^2(\theta)} \frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3 \sin^2(\theta)}\right) \frac{\partial f}{\partial p_r} - \left(\frac{\partial \Phi}{\partial \theta} - \frac{p_\phi^2 \cos(\theta)}{r^2 \sin^3(\theta)}\right) \frac{\partial f}{\partial p_\theta} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} = 0$$

Cylindrical coordinates

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2 \dot{\phi} = R V_\phi \\ p_z = \dot{z} = v_z \end{cases} \qquad H = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right) + \Phi(R, \phi, z)$$

$$\frac{\partial f}{\partial t} + p_R \frac{\partial f}{\partial R} + \frac{p_\phi}{R^2} \frac{\partial f}{\partial \phi} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right) \frac{\partial f}{\partial p_R} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial p_z} = 0$$

Limits of the Collisionless Boltzmann equation

I. Finite stellar lifetime

- Stars are created and die. The hypothesis of conservation of the probability/number is violated.

  We should better have (in Cartesian coordinates):

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = B(\vec{x}, \vec{v}, t) - D(\vec{x}, \vec{v}, t)$$

Rate per unit phase-space volume at which stars are born and die

$$\sim \frac{v}{R} f \qquad \sim \frac{a}{v} f$$

$$\sim \frac{1}{t_{\rm cross}} f \qquad \sim \frac{1}{t_{\rm cross}} f$$

- Define

$$\gamma = \frac{|B - D|}{f} t_{\rm cross}$$

If $\gamma \ll 1$ the approximation is ok

i.e. : the fractional change in the number of stars per crossing time must be small.

Limits of the Collisionless Boltzmann equation

T_{cross} ~= 300 Myr

Examples:

- M-stars in an elliptical galaxies
  - Life time > 10 Gyr (> t_{cross}) $\quad \gamma \cong 0$
  - B=0 (no star formation)

- O-stars in the Milky Way
  - Life time < 100 Myr (< t_{cross}) $\quad \gamma \gg 1$
  - Do not move much, the phase space distribution will be dominated by star formation processes

- Main sequence stars ($M<1.5M_o$)
  - Life time > 1 Gyr (> t_{cross}) $\quad \gamma \cong 0$

Limits of the Collisionless Boltzmann equation

II. Correlation between stars

- We assumed that the probability of finding one peculiar stars somewhere in the phase space is independent of the others. Mathematically: the probability of finding particle "i" in $d^6\vec{\omega}$ and "j" in $d^6\vec{\omega'}$ is :

$$f(\vec{\omega})d^6\vec{\omega} \cdot f(\vec{\omega'})d^6\vec{\omega'}$$

This is not completely true, as stars interact gravitationally and my generate correlations.

However, this is not a real problem as long as the forces between nearby stars do not dominates over the forces due to the rest of the system (the definition of a collisionless system).

Equilibria of collisionless systems

Relations between the DFs and observables

Relations between the DF and observables

$$f(\vec{w})$$

- $f(\vec{w})$: probability density in the phase space
- $f(\vec{w})\, d^6\vec{w}$: probability of finding 1 star in the phase space volume $[\vec{w}, \vec{w} + d\vec{w}]$

Distribution function in the configuration space

$$\nu(\vec{x}) = \int d^3\vec{v}\; f(\vec{x},\vec{v})$$

- $\nu(\vec{x})$: probability density in the configuration space
- $\nu(\vec{x})\,d^3\vec{x}$: probability of finding 1 star in the configuration space volume $[\vec{x}, \vec{x}+d\vec{x}]$

Distribution function in the configuration space

$$n(\vec{x}) = N\nu(\vec{x}) = \int d^3\vec{v}\ \tilde{f}(\vec{x},\vec{v})$$

- $n(\vec{x})$: number density of star in the configuration space
- $n(\vec{x})\, d^3\vec{x}$: probability of finding N stars in the configuration space volume $[\vec{x}, \vec{x}+d\vec{x}]$

Distribution function in the configuration space

$$\rho(\bar{x}) = N \cdot m \cdot \nu(\bar{x}) = m \int d^3\hat{v}\ \tilde{f}(\bar{x}, \hat{v})$$

m : mass of particles

- $\rho(\bar{x})$: mass density of star in the configuration space
- $\rho(\bar{x})\, d^3\bar{x}$: probability of finding a mass $M = N \cdot m$ in the configuration space volume $[\bar{x}, \bar{x} + d\bar{x}]$

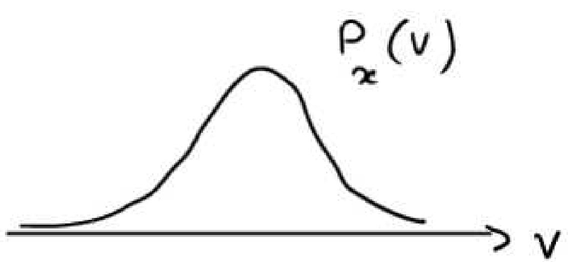
Distribution function in the velocity space

$$P_{\vec{x}}(\vec{v}) = \frac{f(\vec{x}, \vec{v})}{\nu(\vec{x})}$$

$$\int P_{\vec{x}}(\vec{v})\, d^3\vec{v} = \frac{1}{\nu(\vec{x})} \underbrace{\int f(\vec{x}, \vec{v})\, d^3\vec{v}}_{:= \nu(\vec{x})} = 1$$

$\equiv$ velocity distribution function (VDF)

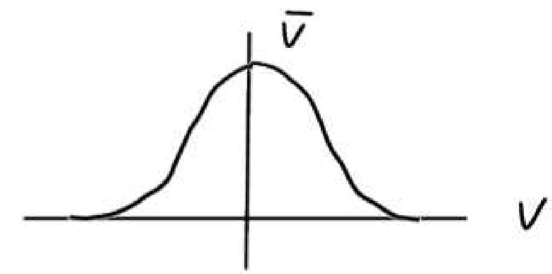
- $P_{\vec{x}}(\vec{v})$: probability density at the position $\vec{x}$ in the velocity space
- $P_{\vec{x}}(\vec{v})\, d^3\vec{v}$: probability of finding 1 star in $\vec{x}$ in the velocity space volume $[\vec{v}, \vec{v} + d\vec{v}]$



Can be measured near the sun

Mean velocity (first moment of the VDF)

$$\vec{\bar{v}}(\bar{x}) = \int \vec{v}\, P_x(\vec{v})\, d^3\vec{v} = \frac{1}{\nu(\bar{x})} \int \vec{v}\, f(\bar{x},\bar{v})\, d^3\vec{v}$$



- along one peculiar axis $\vec{n}$

$$\bar{v}_{\vec{n}}(\bar{x}) = \int \vec{v}\cdot\vec{n}\, P_x(\vec{v})\, d^3\vec{v} = \frac{1}{\nu(\bar{x})} \int \vec{v}\cdot\vec{n}\, f(\bar{x},\bar{v})\, d^3\vec{v}$$

- if $\vec{n} = \vec{e_i}$

$$\bar{v}_i(\bar{x}) = \int v_i\, P_x(\vec{v})\, d^3\vec{v} = \frac{1}{\nu(\bar{x})} \int v_i\, f(\bar{x},\bar{v})\, d^3\vec{v}$$

Velocity dispersion tensor (second moment of the VDF)

$$\sigma_{ij}^2 = \int (v_i - \bar{v}_i)(v_j - \bar{v}_j) \; P_{\vec{x}}(\vec{v}) \, d^3\vec{v}$$

$$= \frac{1}{\nu(\vec{x})} \int (v_i - \bar{v}_i)(v_j - \bar{v}_j) \, f(\vec{x}, \vec{v}) \, d^3\vec{v}$$

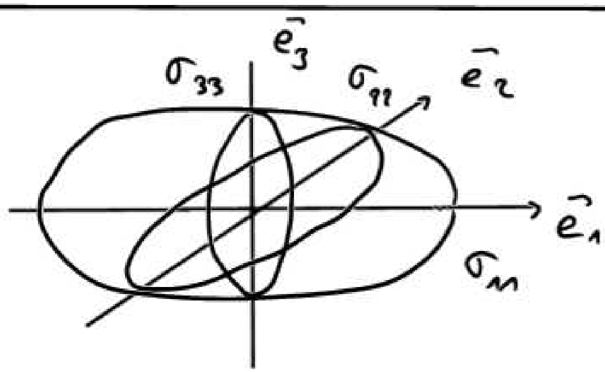
$$= \int v_i v_j \, f(\vec{x},\vec{v}) \, d^3\vec{v} - \left(\int v_i f(\vec{x},\vec{v}) \, d^3v \right)\left(\int v_j f(\vec{x},\vec{v}) \, d^3v \right)$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

3 x 3 symmetric tensor

$\Rightarrow$ may be diagonalised

Describe an ellipsoid (velocity ellipsoid)



$$\sigma_{ij}^2 = \sigma_{ii}^2 \, \delta_{ij} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

Equilibria of collisionless systems

The Jeans Theorems

Question :

> How can we obtain a steady-state solution of the collision-less Boltzmann equation? $\frac{\partial f}{\partial t} = 0$

$$\underbrace{\frac{\partial H}{\partial p}}_{\dot{q}} \frac{\partial f}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p} = 0$$

In carthesian coordinates $\frac{\partial H}{\partial \bar{x}} = \frac{\partial \phi}{\partial \bar{x}}$

$$\frac{\partial f}{\partial \bar{x}} v - \frac{\partial \phi}{\partial \bar{x}} \frac{\partial f}{\partial \bar{v}} = 0$$

Back to the integrals of motion

The function $I(\vec{x}(t), \vec{v}(t))$ is an integral of motion if

$$\frac{d}{dt} I(\vec{x}(t), \vec{v}(t)) = 0$$

along the trajectory.

But

$$\frac{dI}{dt} = \frac{\partial I}{\partial \vec{x}} \dot{\vec{x}} + \frac{\partial I}{\partial \vec{v}} \dot{\vec{v}} = 0$$

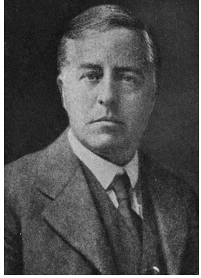
$$= \frac{\partial I}{\partial \vec{x}} \vec{v} - \frac{\partial I}{\partial \vec{v}} \vec{\nabla} \phi = 0$$

Similar to the Collisionless Boltzmann equation

If $I(\vec{x}, \vec{v})$ is an integral of motion

$I(\vec{x}, \vec{v})$ is a steady state solution of the Collisionless Boltzbann equation

James Jeans
1877-1947



Jeans theorems

I. Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion.

II. Any function of integrals of motion yields a steady-state solution of the collisonless Boltzmann equation.

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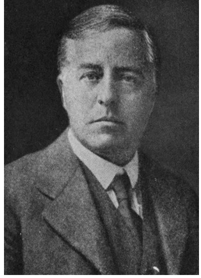

Jeans theorems

I. Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion.

Demonstration:

If a function is a solution of the steady-state collisionless Boltzmann equation, then, it is an integral of motion, thus the function depends on the phase-space coordinates only through integrals of motion (itself !).

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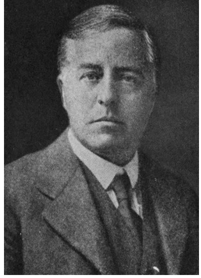

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Demonstration:

Assume $$f(\vec{x},\vec{v}) = f(I_1(\vec{x},\vec{v}), I_2(\vec{x},\vec{v}), I_3(\vec{x},\vec{v}), ...)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{x},\vec{v}) = \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_1}{\mathrm{d}t}}_{=0} + \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_2}{\mathrm{d}t}}_{=0} + \frac{\partial f}{\partial I_1}\underbrace{\frac{\mathrm{d}I_3}{\mathrm{d}t}}_{=0} + ... = 0$$

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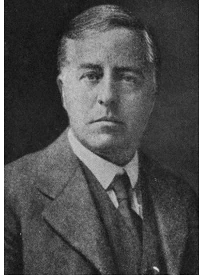

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Extremely useful to generate DFs

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