

Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

Using the definition

$$v_c = R\Omega(R)$$

it follows that

$$\frac{d\Omega}{dR} = \frac{1}{R} \frac{dv_c}{dR} - v_c \frac{1}{R^2}$$

Then

$$A(R) \equiv \frac{1}{2} \left(\frac{v_c}{R} - \frac{dv_c}{dR} \right) = \frac{1}{2} \left(-R \left(\frac{1}{R} \frac{dv_c}{dR} - \frac{v_c}{R^2} \right) \right) = -\frac{1}{2} R \frac{d\Omega}{dR}$$

$$B(R) \equiv -\frac{1}{2} \left(\frac{v_c}{R} + \frac{dv_c}{dR} \right) = -\frac{1}{2} \left(\Omega + \left(\frac{1}{R} \frac{dv_c}{dR} - \frac{v_c}{R^2} \right) + \frac{v_c}{R} \right) = -\left(\Omega + \frac{1}{2} R \frac{d\Omega}{dR} \right)$$

$$\Omega = A - B = \frac{1}{2} \left(\frac{v_c}{R} - \frac{dv_c}{dR} \right) + \frac{1}{2} \left(\frac{v_c}{R} + \frac{dv_c}{dR} \right) = \frac{v_c}{R} = \Omega$$

$$\begin{aligned} \kappa^2 &= \left(R \frac{d(\Omega^2)}{dR} + 4\Omega^2 \right) = \left(2R\Omega \frac{d\Omega}{dR} + 4\Omega^2 \right) = 2\Omega \left(R \frac{d\Omega}{dR} + 2\Omega \right) \\ &= 2\Omega (-2B) = -4B(A - B) \end{aligned}$$

Problem 2:

We have:

$$R^2 = x^2 + y^2$$

$$\vec{L} = \vec{r} \times \vec{v} = \vec{x} \times \dot{\vec{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{pmatrix}$$

Since we're working in the $z = 0$ plane, and the z component of $\vec{L}$ is given by $L_z = x\dot{y} - y\dot{x}$, inserting this to compute L^2 gives

$$L^2 = (y\dot{z} - z\dot{y})^2 + (z\dot{x} - x\dot{z})^2 + (x\dot{y} - y\dot{x})^2 = (x^2 + y^2)\dot{z}^2 + L_z^2 = R^2\dot{z}^2 + L_z^2$$

Now we use the energy conservation:

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{eff}(R, z)$$

Now eliminate $\dot{z}^2$ by using our expression for L^2 :

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{1}{R^2}(L^2 - L_z^2) + \Phi_{eff}(R, z)$$

Solving for $\dot{R}$ gives and using $\Phi_{eff} = \frac{1}{2}\frac{L_z^2}{R^2} + \Phi$:

$$\dot{R} = \pm \sqrt{2 \left(E - \frac{1}{2} \frac{L^2 - L_z^2}{R^2} - \Phi_{eff} \right)} = \pm \sqrt{2 \left(E - \frac{L^2}{2R^2} - \Phi \right)}$$

Problem 3:

We start from the Lagrangian:

$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} \left(\dot{\vec{x}} + \Omega \times \vec{x} \right)^2 - \Phi(\vec{x}) \quad (1)$$

From the derivative of this Lagrangian, we can write the momentum $\vec{p}$:

$$\vec{p} = \dot{\vec{x}} + \Omega \times \vec{x} \quad (2)$$

Using the Legendre transformation, we obtain the Hamiltonian that writes:

$$H(q, p) = \frac{1}{2}\vec{p}^2 + \Phi(\vec{q}) - \vec{\Omega} \cdot (\vec{q} \times \vec{p}), \quad (3)$$

where we renamed $\vec{x}$ by $\vec{q}$.

We set the rotation to be along the z axis, and for it to be uniformly rotating, it needs to be constant, i.e.

$$\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \Rightarrow \vec{\Omega} \cdot (\vec{q} \times \vec{p}) = \Omega(q_x p_y - p_x q_y)$$

The equations of motion in canonical coordinates are given by Hamilton's equations:

$$\dot{p} = -\frac{\partial}{\partial q} H(p, q), \quad \dot{q} = \frac{\partial}{\partial p} H(p, q) \quad (4)$$

in our case:

$$\begin{aligned} \dot{q}_x &= p_x + \Omega q_y \\ \dot{q}_y &= p_y - \Omega q_x \\ \dot{p}_x &= -\frac{\partial}{\partial q_x} \Phi(q, p) + \Omega p_y \\ \dot{p}_y &= -\frac{\partial}{\partial q_y} \Phi(q, p) - \Omega p_x \end{aligned}$$

The relations between cartesian and canonical coordinates are:

$$q_x = x$$

$$q_y = y$$

$$p_x = \dot{x} - \Omega y$$

$$p_y = \dot{y} + \Omega x$$