Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

Write the definition of the specific angular momentum in cylindrical coordinates:

$$\vec{L} = \vec{x} \times \vec{v} = (R\vec{e}_R + z\vec{e}_z) \times \left(\dot{R}\vec{e}_R + R\dot{\theta}\vec{e}_\theta + \dot{z}\vec{e}_z\right) \tag{1}$$

$$= \left(-zR\dot{\theta}\right)\vec{e}_R + \left(z\dot{R} - R\dot{z}\right)\vec{e}_\theta + \left(R^2\dot{\theta}\right)\vec{e}_z \tag{2}$$

So, we have:

$$L_R = -zR\dot{\theta} \tag{3}$$

$$L_\theta = z\dot{R} - R\dot{z} \tag{4}$$

$$L_z = R^2\dot{\theta} \tag{5}$$

We can now write:

$$\dot{L}_R - L_\theta\dot{\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-zR\dot{\theta}\right) - \left(z\dot{R} - R\dot{z}\right)\dot{\theta} = 0 \tag{6}$$

$$= -\dot{z}R\dot{\theta} - z\dot{R}\dot{\theta} - zR\ddot{\theta} - z\dot{R}\dot{\theta} + \dot{z}R\dot{\theta} \tag{7}$$

So, this implies that:

$$2\dot{R}\dot{\theta} + R\ddot{\theta} = 0 \tag{8}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2\dot{\theta}\right) = 0 = \frac{\mathrm{d}}{\mathrm{d}t}L_z \tag{9}$$

and thus L_z being constant.

Problem 2:

For a given potential $\Phi(\vec{r})$, the acceleration reads:

$$\vec{a}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) \tag{10}$$

General cartesian:
$\Phi(\vec{r}) = \Phi(x,y,z)$; $\vec{\nabla}\Phi(\vec{r}) = \frac{\partial\Phi(x,y,z)}{\partial x}\vec{e_x} + \frac{\partial\Phi(x,y,z)}{\partial y}\vec{e_y} + \frac{\partial\Phi(x,y,z)}{\partial z}\vec{e_z}$.

Spherical symmetry:

$\Phi(\vec{r}) = \Phi(r)$; $\vec{\nabla}\Phi(\vec{r}) = \frac{d\Phi(r)}{dr}\vec{e_r}$, and $\vec{e_r} = \frac{x}{r}\vec{e_x} + \frac{y}{r}\vec{e_y} + \frac{z}{r}\vec{e_z}$.

$$\begin{aligned}
a_x &= -\frac{d\Phi(r)}{dr} \times \frac{x}{r} && (11)\\
a_y &= -\frac{d\Phi(r)}{dr} \times \frac{y}{r} && (12)\\
a_z &= -\frac{d\Phi(r)}{dr} \times \frac{z}{r} && (13)\\
& && (14)
\end{aligned}$$

Cylindrical symmetry:
$\Phi(\vec{r}) = \Phi(R, z)$; $\vec{\nabla}\Phi(R, z) = \frac{\partial\Phi(R,z)}{\partial r}\vec{e_r} + \frac{\partial\Phi(R,z)}{\partial z}\vec{e_z}$, and $\vec{e_r} = \frac{x}{R}\vec{e_x} + \frac{y}{R}\vec{e_y}$.

$$\begin{aligned}
a_x &= -\frac{\partial\Phi(R, z)}{\partial R} \times \frac{x}{R} && (15)\\
a_y &= -\frac{\partial\Phi(R, z)}{\partial R} \times \frac{y}{R} && (16)\\
a_z &= -\frac{\partial\Phi(R, z)}{\partial z} && (17)
\end{aligned}$$

The derivative of the potentials are the following:

a) Point mass:

$$\frac{d\Phi(r)}{dr} = \frac{GM}{r^2} \tag{18}$$

b) Plummer-Schuster potential:

$$\frac{d\Phi(r)}{dr} = \frac{GMr}{(r^2 + e^2)^{3/2}} \tag{19}$$

c) Miyamoto-Nagai potential:

$$\frac{\partial \Phi(R,z)}{\partial R} = \frac{GMR}{\left(R^2 + \left[a + \sqrt{z^2+b^2}\right]^2\right)^{3/2}} \tag{20}$$

$$\frac{\partial \Phi(R,z)}{\partial z} = \frac{GMz}{\left(R^2 + \left[a + \sqrt{z^2+b^2}\right]^2\right)^{3/2}} \left(1 + \frac{a}{\sqrt{z^2+b^2}}\right) \tag{21}$$

$$\tag{22}$$

d) Harmonic potential:

$$\frac{\partial \Phi(x,y,z)}{\partial x} = \omega_x^2 x \tag{23}$$

$$\frac{\partial \Phi(x,y,z)}{\partial y} = \omega_y^2 y \tag{24}$$

$$\frac{\partial \Phi(x,y,z)}{\partial z} = \omega_z^2 z \tag{25}$$

The circular velocity follows trivially :

$$V_c = \sqrt{R \frac{\partial}{\partial R} \Phi(R)} \tag{26}$$

So does the period :

$$T = 2\pi \frac{R}{V_c(R)} \tag{27}$$

Problem 3:

For a given potential in cylindrical coordinates $\Phi(R, \theta, z)$, the vertical and epicyclic frequencies read:

$$\Omega^2(R_g) = \frac{1}{R} \left(\frac{\partial \Phi}{\partial R}\right)_{(R_g, z=0)} \tag{28}$$

$$\nu^2(R_g) = \left(\frac{\partial^2 \Phi}{\partial z^2}\right)_{(R_g, z=0)} \tag{29}$$

$$\kappa^2(R_g) = \left(\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right)_{(R_g, z=0)} = \left(R \frac{\partial (\Omega^2)}{\partial R} + 4\Omega^2\right)_{(R_g, z=0)} \tag{30}$$

For the point mass, one obtains (dropping the g subscript for short):

$$\begin{aligned}
\Omega^2(R) &= \frac{GM}{R^3} && (31)\\
\nu^2(R) &= \frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial z}\right)_{(R_g,z=0)} = \frac{\partial}{\partial z}\left(\frac{GMz}{(x^2+y^2+z^2)^{3/2}}\right)_{(R_g,z=0)} && (32)\\
&= \frac{GM}{R^3} - 3\left(\frac{GMz^2}{R^5}\right)_{z=0} = \frac{GM}{R^3} && (33)\\
\kappa^2(R) &= \left[\frac{\partial^2\Phi}{\partial R^2} - \frac{1}{R}\frac{\partial\Phi}{\partial R} + \frac{4}{R}\frac{\partial\Phi}{\partial R}\right]_{(R_g,z=0)} = \frac{GM}{R^3} && (34)
\end{aligned}$$

For the Plummer-Schuster potential:

$$\begin{aligned}
\Omega^2(R) &= \frac{GM}{(e^2+R^2)^{3/2}} && (35)\\
\nu^2(R) &= = \frac{GM}{(e^2+R^2)^{3/2}} && (36)\\
\kappa^2(R) &= 4\frac{GM}{(e^2+R^2)^{3/2}} - 3R^2\frac{GM}{(e^2+R^2)^{5/2}} && (37)
\end{aligned}$$

For the Miyamoto-Nagai potential:

$$\begin{aligned}
\Omega^2(R) &= \frac{GM}{(R^2+(a+b)^2)^{3/2}} && (38)\\
\nu^2(R) &= \frac{GM(a+b)}{b(R^2+(a+b)^2)^{3/2}} && (39)\\
\kappa^2(R) &= 4\frac{GM}{(R^2+(a+b)^2)^{3/2}} - 3R^2\frac{GM}{(R^2+(a+b)^2)^{5/2}} && (40)
\end{aligned}$$